**MATH 553 QUALIFYING EXAMINATION**

**January 2010**

Please begin each question I–V on a new sheet.

In doing any part of a multipart problem, you may assume you’ve done the preceding parts, even if you haven’t.

**I. 32 points** Let $p$ and $q$ be (positive) integer primes such that $p$ divides $q - 1$.

(a) Show that there exists a group $G$ of order $p^2q$ with generators $x$ and $y$ such that $x^{p^2} = 1$, $y^q = 1$, and $x^y = x^{a/q}$, with $1$ the identity element and a some integer such that $a \not\equiv 1 \pmod{q}$ but $a^p \equiv 1 \pmod{q}$.

(b) Prove that the Sylow $q$-subgroup $S_q \subseteq G$ is normal.

(c) Prove that $G/S_q$ is cyclic; and deduce that $G$ has a unique subgroup $H$ of order $pq$.

(d) Prove that $H$ is cyclic.

(e) Prove that any order-$p$ subgroup of $G$ is contained in $H$, hence is generated by $x^p$ and is contained in the center of $G$.

(f) Prove that the center of $G$ is the unique order-$p$ subgroup of $G$.

(g) Prove that every subgroup of $G$ other than $G$ itself is cyclic.

(h) For each divisor $d$ of $p^2q$, say how many elements of order $d$ there are in $G$.

**II. 33 points**

(a) Prove that the ring $R = \mathbb{Z}[\sqrt{-2}]$ is Euclidean.

(b) Show that $R/(3 + 2\sqrt{-2}) \cong \mathbb{F}_{17}$, the field with 17 elements.

(c) Show that the polynomial $X^4 + 3$ is irreducible over the field $\mathbb{F}_{17}$, and deduce that the polynomial $f(X) := X^4 - 170X^3 + 9 + 4\sqrt{-2} \in R[X]$ is irreducible.

(d) Is the polynomial $Y^4 - f(X) \in R[X,Y]$ irreducible? (Why?)

**III. 8 points** Prove or disprove: If $E \subseteq F \subseteq G$ are fields such that $F$ is a finite Galois extension of $E$ and $G$ is a finite Galois extension of $F$, then $G$ is a finite Galois extension of $E$.

**IV. 12 points** Let $E$ be a field and let $F$ be a finite Galois extension of $E$. Let $h(X)$ be an irreducible monic polynomial in $E[X]$, and let $h_1(X)$, $h_2(X)$ be two irreducible monic polynomials in $F[X]$ both of which divide $h(X)$. Then (prove): there exists an automorphism $\theta$ of $F[X]$ such that $\theta$ leaves all elements in $E[X]$ fixed and furthermore $\theta(h_1) = h_2$.

**V. 15 points** Let $k$ be a commutative field, and let $k(X)$ be the field of fractions of the polynomial ring $k[X]$. Let $f$ and $g$ be the unique automorphisms of $k(X)$ fixing $k$ and such that

$$f(X) = 1/X, \quad g(X) = 1 - X.$$  

In the group of all automorphisms of $k(X)$, let $G$ be the subgroup generated by $f$ and $g$.

(a) Write down explicitly all the members of $G$. ($f$ and $g$ are already given above; specify the other members similarly.)

(b) Show that the fixed field of $G$ is $k(Y)$, where

$$Y = (X^2 - X + 1)^3/X^2(X - 1)^2.$$  

Hint. $X$ is a root of the sixth-degree polynomial $(T^2 - T + 1)^3 - Y(T^2)(T - 1)^2 \in k(Y)[T]$.

(c) Show: If $k(Y) \not\subseteq L \not\subseteq k(X)$ with $L/k(Y)$ a normal field extension, then $L = k(Z)$ where

$$Z = X + (1 - 1/X) + \frac{1}{1 - X}.$$