MATH 553 QUALIFYING EXAMINATION
January 2010

PLEASE BEGIN EACH QUESTION I–V ON A NEW SHEET.

IN DOING ANY PART OF A MULTIPART PROBLEM, YOU MAY ASSUME YOU'VE DONE THE PRECEDING PARTS, EVEN IF YOU HAVEN'T.

I. [32 points] Let p and q be (positive) integer primes such that p divides $q-1$.

(a) Show that there exists a group G of order p^2q with generators x and y such that $x^{p^2}=1$, $y^q=1$, and $xyx^{-1}=y^a$, with 1 the identity element and a some integer such that $a\not\equiv 1 \pmod q$ but $a^p\equiv 1 \pmod q$.

(b) Prove that the Sylow q-subgroup $S_q\subset G$ is normal.

(c) Prove that G/S_q is cyclic; and deduce that G has a unique subgroup H of order pq.

(d) Prove that H is cyclic.

(e) Prove that any order-p subgroup of G is contained in H, hence is generated by x^p and is contained in the center of G.

(f) Prove that the center of G is the unique order-p subgroup of G.

(g) Prove that every subgroup of G other than G itself is cyclic.

(h) For each divisor d of p^2q, say how many elements of order d there are in G.

II. [33 points] (a) Prove that the ring $R=\mathbb{Z}[\sqrt{-2}\,]$ is Euclidean.

(b) Show that $R/(3+2\sqrt{-2}\,)\cong\mathbb{F}_{17}$, the field with 17 elements.

(c) Show that the polynomial X^4+3 is irreducible over the field $\mathbb{F}_{17}$, and deduce that the polynomial $f(X):=X^4-170X^3+9+4\sqrt{-2}\in R[X]$ is irreducible.

(d) Is the polynomial $Y^4-f(X)\in R[X,Y]$ irreducible? (Why?)

III. [8] Prove or disprove: If $E\subseteq F\subseteq G$ are fields such that F is a finite Galois extension of E and G is a finite Galois extension of F, then G is a finite Galois extension of E.

IV. [12] Let E be a field and let F be a finite Galois extension of E. Let $h(X)$ be an irreducible monic polynomial in $E[X]$, and let $h_1(X)$, $h_2(X)$ be two irreducible monic polynomials in $F[X]$ both of which divide $h(X)$. Then (prove): there exists an automorphism θ of $F[X]$ such that θ leaves all elements in $E[X]$ fixed and furthermore $\theta(h_1)=h_2$.

V. [15] Let k be a commutative field, and let $k(X)$ be the field of fractions of the polynomial ring $k[X]$. Let f and g be the unique automorphisms of $k(X)$ fixing k and such that

$$f(X)=1/X, \qquad g(X)=1-X.$$

In the group of all automorphisms of $k(X)$, let G be the subgroup generated by f and g.

(a) Write down explicitly all the members of G. (f and g are already given above; specify the other members similarly.)

(b) Show that the fixed field of G is $k(Y)$, where

$$Y=(X^2-X+1)^3/X^2(X-1)^2.$$

Hint. X is a root of the sixth-degree polynomial $(T^2-T+1)^3-Y(T^2)(T-1)^2\in k(Y)[T]$.

(c) Show: If $k(Y)\subsetneqq L\subsetneqq k(X)$ with $L/k(Y)$ a normal field extension, then $L=k(Z)$ where

$$Z=X+(1-\frac{1}{X})+\frac{1}{1-X}\,.$$