MATH 553 QUALIFYING EXAM
AUGUST 2011

Instructions: Give a complete solution to each problem. Be sure to show all your work. You may cite any result except the one you are asked to prove. If a result has a name, you may refer to it by name. Otherwise, be sure to indicate the content of the result. The exam is graded 0-200 points.

1. Let

$$G = GL_2(\mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, \text{ and } ad - bc \not\equiv 0 \pmod{p} \right\}.$$

(a) **(12 points)** Find the order, $|G|$, of G.

(b) **(10 points)** Show $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in \mathbb{Z}/p\mathbb{Z} \right\}$ is a Sylow p-subgroup of G.

(c) **(12 points)** Find the number of Sylow p–subgroups of G.

2. (11 points each)

(a) Let G be a group, H a subgroup of G with $[G : H] = 2$. Suppose K is a subgroup of G of odd order. Show $K \subseteq H$.

(b) Let G be a finite group and suppose there is a sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = H,$$

with $[G_i : G_{i+1}] = 2$ for $i = 1, 2, , \ldots n-1$. Suppose H has odd order. Show $H \triangleleft G$.

(c) Suppose $|G| = 2^n m$, with m odd. Suppose G has a normal subgroup H of order m. Show there is a sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_n = H$, with $[G_i : G_{i+1}] = 2$, for all i.

3. (12 points each) Let R be a commutative ring with identity $1 \neq 0$, and let I be an ideal of R. Define $\operatorname{Jac} I$ to be the intersection of all maximal ideals containing I, with the convention $\operatorname{Jac} R = R$. Let $\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n > 0\}$.

(a) Prove $\operatorname{Jac} I$ is an ideal of R containing I.

(b) Prove $\sqrt{I} \subseteq \operatorname{Jac} I$.

(c) Let F be a field, set $R = F[x]$, and let $I = (f(x))$, for some **non-zero** $f(x) \in R$. Describe $\operatorname{Jac} I$ in this instance.

4. Let S be the subring of $\mathbb{C}[x] \times \mathbb{C}[y]$ consisting of pairs (f, g) with $f(0) = g(0)$.
   (a) **(12 points)** Let $\varphi : \mathbb{C}[x, y] \to S$ be defined by $\varphi(h) = (f, g)$, where $f(x) = h(x, 0)$, and $g(y) = g(0, y)$. Prove φ is a surjective homomorphism.
   (b) **(10 points)** Prove $\mathbb{C}[x, y]/(xy) \simeq S$.
   (c) **(10 points)** Use (b) to describe the prime ideals of S. Be sure to justify your answer.
5. Let p be a prime. let $F = \mathbb{F}_p$ be the field of p elements and $K = \mathbb{F}_{p^{10}}$ be the unique extension of F with p^{10} elements.
   (a) **(8 points)** Find all subfields of K. Make sure to justify your answer.
   (b) **(16 points)** Find a formula for the number of monic irreducible polynomials of degree 10 in $F[x]$. Justify your answer.

6. Let $f(x) = (x^2 - 3)(x^3 - 7) \in \mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$.
   (a) **(14 points)** Find the degree of K over $\mathbb{Q}$.
   (b) **(16 points)** Classify the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
   (c) **(11 points)** Find all subfields E of K so that $E/\mathbb{Q}$ is a quadratic extension.