Note. There are 6 questions in 14 parts. Each part worths 9 points, except Problem 6 worths 18 points. But the maxmal points you may receive is 100 points. You may do them in any order but label your solutions clearly. You may do any problem/part by assuming the conclusion of other problems/parts.

1. Let $A_7$ be the alternating group of order $7!/2 = 504 \times 5$. Show that $A_7$ has no subgroup of order 504. You can freely use the fact that $A_7$ is simple.

2. Let $q = p^f > 1$ be an integral power of a prime number $p$. Denote by $F_q$ a finite field of cardinality $q$. Let $G = \text{GL}_2(F_q)$ be the group of $2 \times 2$ invertible matrices with coefficients in $F_q$. You can take for granted that the cardinality of $G$ is $(q^2 - 1)(q^2 - q)$.

   (a) Let $H = \{g \in G : \det(g) = 1\}$. Show that $H$ is a subgroup of order $(q^2 - 1)q$.
   (b) Show that $H$ contains a subgroup isomorphic to $F_q^*$, the multiplicative group of units in $F_q$.
   (c) Show that $G$ contains a subgroup isomorphic to $F_q^{*2}$.
   (d) Let $\ell$ be a prime such that $\ell \neq 2$, $\ell \neq p$. Show that the Sylow $\ell$-subgroup of $H$ is cyclic.
   (e) Show that the Sylow $p$-subgroup of $H$ is cyclic exactly when $f = 1$.
   (f) Assume that $p$ is odd. Show that the Sylow 2-subgroup of $H$ is not cyclic.

3. Let $R$ and $S$ be commutative rings with 1. Let $I$ and $J$ be ideals of $R$ and $S$ respectively.

   (a) Show that $I \times J$ is an ideal of $R \times S$.
   (b) Show that every ideal of $R \times S$ is of the form $I \times J$ for some $I, J$ as above.
   (c) Show that $I \times J$ is a prime ideal if and only if $I$ is a prime ideal and $J = S$, or $I = R$ and $J$ is a prime ideal.

4. Let $k$ be a field and let $E = k(t) := \text{Frac} k[t]$ be the field of fractions in an indeterminate $t$. Let $\sigma \in \text{Aut}(E/k)$ be the automorphism characterized by $\sigma(t) = t + 1$.

   (a) Show that if the characteristic of $k$ is zero, then $E^\sigma = k$.
   (b) Show that if the characteristic of $k$ is $p > 0$, then $E^\sigma = k(t^p - t)$.

5. Show that the polynomial $f(X) = X^4 + aX^3 + bX^2 + aX + 1$ is irreducible over $\mathbb{C}[a, b]$, where $a, b$ are indeterminates over $\mathbb{C}$.

6. Let $F$ be a field of characteristic 0 and let $a \in F^\times$. Exhibit all possible Galois groups of $f(X) = X^4 - a$ over $F$ (up to conjugacy of subgroups of $S_4$). Exhibit one pair of $(F, a)$ for each possibility and show that there is no other possibility.