MATH 553 Qualifying Exam August 2012

Note. There are 6 questions in 14 parts. Each part worths 9 points, except Problem **6** worths 18 points. But the maxmal points you may receive is 100 points. You may do them in any order but label your solutions clearly. You may do any problem/part by assuming the conclusion of other problems/parts.

1. Let A_7 be the alternating group of order $7!/2 = 504 \times 5$. Show that A_7 has no subgroup of order 504. You can freely use the fact that A_7 is simple.

2. Let $q = p^f > 1$ be an integral power of a prime number p. Denote by $\mathbb{F}_q$ a finite field of cardinality q. Let $G = \mathrm{GL}_2(\mathbb{F}_q)$ be the group of 2×2 invertible matrices with coefficients in $\mathbb{F}_q$. You can take for granted that the cardinality of G is $(q^2 - 1)(q^2 - q)$.

(a) Let $H = \{g \in G : \det(g) = 1\}$. Show that H is a subgroup of order $(q^2 - 1)q$.

(b) Show that H contains a subgroup isomorphic to $\mathbb{F}_q^\times$, the multiplicative group of units in $\mathbb{F}_q$.

(c) Show that G contains a subgroup isomorphic to $\mathbb{F}_{q^2}^\times$.

(d) Let ℓ be a prime such that $\ell \neq 2$, $\ell \neq p$. Show that the Sylow ℓ-subgroup of H is cyclic.

(e) Show that the Sylow p-subgroup of H is cyclic exactly when $f = 1$.

(f) Assume that p is odd. Show that the Sylow 2-subgroup of H is not cyclic.

3. Let R and S be commutative rings with 1. Let I and J be ideals of R and S respectively.

(a) Show that $I \times J$ is an ideal of $R \times S$.

(b) Show that every ideal of $R \times S$ is of the form $I \times J$ for some I, J as above.

(c) Show that $I \times J$ is a prime ideal if and only if I is a prime ideal and $J = S$, or $I = R$ and J is a prime ideal.

4. Let k be a field and let $E = k(t) := \mathrm{Frac}\, k[t]$ be the field of fractions in an indeterminate t. Let $\sigma \in \mathrm{Aut}(E/k)$ be the automorphism characterized by $\sigma(t) = t + 1$.

(a) Show that if the characteristic of k is zero, then $E^\sigma = k$.

(b) Show that if the characteristic of k is $p > 0$, then $E^\sigma = k(t^p - t)$.

5. Show that the polynomial $f(X) = X^4 + aX^3 + bX^2 + aX + 1$ is irreducible over $\mathbb{C}[a, b]$, where a, b are indeterminates over $\mathbb{C}$.

6. Let F be a field of characteristic 0 and let $a \in F^\times$. Exhibit all possible Galois groups of $f(X) = X^4 - a$ over F (up to conjugacy of subgroups of S_4). Exhibit one pair of (F, a) for each possibility and show that there is no other possibility.