PUID: ________________________________

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

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1. (20 pts) Let $p$ be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with $p$ elements. Let $V$ be a vector space over $F$ and $T : V \to V$ a linear operator. Assume that $T$ has characteristic polynomial $x^3$ and minimal polynomial $x^2$.

(a) Express $V$ as a direct sum of cyclic $F[x]$-modules.

(b) How many non-cyclic 2-dimensional $T$-invariant subspaces does $V$ have?

(c) How many 2-dimensional $T$-invariant subspaces of $V$ are direct summands of $V$?

(d) How many 1-dimensional $T$-invariant subspaces does $V$ have?

(e) How many 1-dimensional $T$-invariant subspaces of $V$ are not direct summands of $V$?
2. Let $V$ be a finite-dimensional vector space over a field $F$, let $T : V \to V$ be a linear operator, and let $p(x) \in F[x]$ be the minimal polynomial of $T$. Assume that $p(x) = p_1^{r_1} \cdots p_k^{r_k}$, where the $p_i \in F[x]$ are distinct monic irreducible polynomials, $i = 1, \cdots, k$, and the $r_i$ are positive integers. Let $W_i = \{ \alpha \in V \mid p_i(T)^{r_i}(\alpha) = 0 \}$.

(a) (10 pts) Describe how to obtain linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each $i$, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on $V$.

(b) (10 pts) If $p(x)$ is a product of linear polynomials, describe how to obtain a diagonalizable operator $D$ and a nilpotent operator $N$ such that $T = D + N$, where $D$ and $N$ are both polynomials in $T$. 
3. (20 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $T : V \to V$ be a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.

(b) In terms of an expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant subspaces? Explain.
4. (20 pts) Let $V$ be a finite-dimensional vector space over a field $F$ and let $W_1, W_2$ and $W_3$ be nonzero subspaces of $V$.

(a) If $W_1 \cap W_2 = 0$, prove or disprove that every vector $\beta$ in $W_1+W_2$ has a unique representation as $\beta = \alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

(b) If $W_i \cap W_j = 0$ for each $i \neq j$ with $i, j \in \{1, 2, 3\}$, prove or disprove that every vector $\beta$ in $W_1+W_2+W_3$ has a unique representation as $\beta = \alpha_1 + \alpha_2 + \alpha_3$, where $\alpha_i \in W_i$, $1 \leq i \leq 3$. 
5. (20 pts) Let $D$ be a principal ideal domain, let $n$ be a positive integer, and let $D^{(n)}$ denote a free $D$-module of rank $n$.

(a) If $L$ is a submodule of $D^{(n)}$, prove that $L$ is a free $D$-module of rank $m \leq n$.

(b) If $L$ is a proper submodule of $D^{(n)}$, prove or disprove that rank $L < n$. 
6. (15 pts) Let $M$ be a module over an integral domain $D$. A submodule $N$ of $M$ is pure in $M$ if for every $y \in N$ and $a \in D$ the following condition holds: if $ax = y$ for some $x \in M$, then there exists $z \in N$ with $az = y$.

(a) If $M = \langle m \rangle$ is a cyclic $\mathbb{Z}$-module of order 24, list all the pure submodules of $M$.

(b) For a submodule $N$ of $M$ and $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of $x$ in $M/N$. Prove that $\text{ann} \overline{x} := \{a \in D \mid a\overline{x} = 0\} \supseteq \text{ann} x := \{a \in D \mid ax = 0\}$.

(c) If $N$ is pure in $M$, and ann $\overline{x}$ is the principal ideal $(d)$ of $D$, prove that there exists $x' \in M$ such that $x + N = x' + N$ and ann $x' = (d)$. 
7. (13 pts) Let $M$ be a finitely generated module over the polynomial ring $F[x]$, where $F$ is a field, and let $N$ be a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$. 
8. (18 pts) Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional vector space $V$ and let $R = T(V)$ denote the range of $T$.

(a) Prove that $R$ has a complementary $T$-invariant subspace if and only if $R$ is independent of the null space $N$ of $T$, i.e., $R \cap N = 0$.

(b) If $R$ and $N$ are independent, prove that $N$ is the unique $T$-invariant subspace of $V$ that is complementary to $R$. 
9. (20 pts) Let $A$ and $B$ be in $\mathbb{Q}^{n \times n}$ and let $I \in \mathbb{Q}^{n \times n}$ denote the identity matrix.

(a) State true or false and justify: if $A$ and $B$ are similar over an extension field $F$ of $\mathbb{Q}$, then $A$ and $B$ are similar over $\mathbb{Q}$.

(b) Let $M$ and $N$ be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[x]$. Define “$M$ and $N$ are equivalent over $\mathbb{Q}[x]$.”

(c) State true or false and justify: If $\det(xI - A) = \det(xI - B)$, then $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$.

(d) State true or false and justify: If $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$, then $A$ and $B$ are similar over $\mathbb{Q}$.
10. (18 pts) Let $A \in \mathbb{C}^{4 \times 4}$ be a diagonal matrix with exactly three distinct entries on its main diagonal.

(a) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in $A$?

(b) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{4 \times 4}$ such that $AB = BA$?

(c) If $B \in \mathbb{C}^{4 \times 4}$ is a diagonal matrix with exactly three distinct entries on its main diagonal, is $B$ similar to a polynomial in $A$? Justify your answer.
11. (8 pts) Let $V$ be an abelian group with generators $(v_1, v_2, v_3)$ that has the matrix
\[
\begin{bmatrix}
4 & 0 & 8 \\
4 & 12 & 0 \\
\end{bmatrix}
\]
as a relation matrix. Express $V$ as a direct sum of cyclic groups.

12. (8 pts) Let $V$ be an abelian group with generators $(v_1, v_2, v_3)$ that has the matrix
\[
\begin{bmatrix}
4 & 0 & 8 \\
4 & 12 & 0 \\
2 & 2 & 0 \\
\end{bmatrix}
\]
as a relation matrix. Express $V$ as a direct sum of cyclic groups.