Attempt all questions. Time 2 hrs.

1. (5+10+5 pts) Let $E$ be a finite dimensional complex vector space and $u \in \text{End}(E)$.
   (a) Prove that if $\text{Tr}(u^i) = 0$ for each $i > 0$, then $u$ is nilpotent.
   (b) Suppose that
       $$u = [u_1, v_1] + \cdots + [u_m, v_m],$$
       (where for any $f, g \in \text{End}(E)$ we denote $[f, g] = f \circ g - g \circ f$), and $u$ commutes with each $u_i$ for $1 \leq i \leq m$. Prove that $u$ is nilpotent.
   (c) Suppose that $\text{Tr}(u \circ v) = 0$ for all $v \in \text{End}(E)$ satisfying $\text{Tr}(v) = 0$. Prove that $u = \lambda \cdot \text{Id}_E$ for some $\lambda \in k$.

2. (10+10 pts) Let $E$ be a finite dimensional complex inner product vector space and $u, v \in \text{End}(E)$.
   (a) Prove that if $u$ is normal, then $u^* = p(u)$ for some polynomial $p \in \mathbb{C}[X]$.
   (b) Suppose that $u, v \in \text{End}(E)$, such that $u, v$ are normal and $u \circ v = v \circ u$. Prove that $u \circ v$ is normal.

3. (5+15 pts) Let $E$ be a finite dimensional $k$-vector space and $u \in \text{End}(E)$. Consider $\text{End}(E)$ as a $k$-vector space, and denote by $\text{ad}(u)$ the element of $\text{End}(\text{End}(E))$ defined by $\text{ad}(u)(v) = u \circ v - v \circ u$.
   (a) State the additive Jordan decomposition theorem for endomorphisms of finite dimensional complex vector spaces.
   (b) Prove that $\text{ad}(u)_s = \text{ad}(u_s)$, $\text{ad}(u)_n = \text{ad}(u_n)$.
      (using the notation for the additive Jordan decomposition).

4. (5+5+10 pts) Let $E$ be a finite dimensional complex vector space, and $u \in \text{End}(E)$.
   (a) Define the algebraic and geometric multiplicity of an eigenvalue $\lambda$ of $u$.
   (b) What are the algebraic and geometric multiplicities of the various eigenvalues of the endomorphism whose matrix with respect to a certain basis is given by
       $$\begin{pmatrix}
       1 & 0 & 0 & 0 & 0 & 0 \\
       0 & 1 & 0 & 0 & 0 & 0 \\
       0 & 0 & 2 & 0 & 0 & 0 \\
       0 & 0 & 0 & 2 & 0 & 0 \\
       0 & 0 & 0 & 1 & 2 & 0 \\
       0 & 0 & 0 & 0 & 1 & 2
       \end{pmatrix}.$$
   (c) Compute the rational canonical form of the matrix given in Part (4b).

5. (10+10 pts)
   (a) Let $M$ be a $3 \times 3$ matrix with complex entries. If $M^3$ is the identity matrix, what are the possibilities for the Jordan canonical form of $M$?
(b) Let $M$ be a $3 \times 3$ matrix with integer entries and $\det(M) = -1$. Assume that every eigenvalue of $M$ is rational. What are the possibilities for the minimal polynomial and Jordan canonical form of $M$?