In answering any part of a question you may assume the preceding parts.

NOTATION: \( K \) is a field; \( M_n(K) \) is the set of \( n \times n \) matrices with elements from \( K \); \( V \) is an \( n \)-dimensional vector space over \( K \); \( \alpha \) is a linear operator on \( V \).

1. Prove that if \( A, B \in M_n(K) \) and one of \( A, B \) is invertible, then \( \det(aA + B) = 0 \) for at most \( n \) distinct values of \( a \in K \). [8 points]

2. Let \( A^T \) denote the transpose of \( A \in M_n(K) \). Prove that there exists an invertible \( P \in M_n(K) \), such that \( PAP^{-1} = A^T \). [8 points]

3. Let \( \pi_1 \) and \( \pi_2 \) be linear operators on a vector space \( V \), such that
\[
\pi_1 \pi_2 = \pi_2 \pi_1, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2.
\]
Prove that \( V \) is the direct sum of the following four subspaces: [8 points]
\[
\text{Im } \pi_1 \cap \text{Im } \pi_2, \quad \text{Im } \pi_1 \cap \text{Ker } \pi_2, \quad \text{Ker } \pi_1 \cap \text{Im } \pi_2, \quad \text{Ker } \pi_1 \cap \text{Ker } \pi_2.
\]

4. Prove that if \( \alpha \) has the same matrix in all bases of \( V \), then there exists an \( a \in K \) such that \( \alpha = a \text{id}_V \). [8 points]

5. Prove that if \( \text{rank}(\alpha) = 1 \), then the minimal polynomial of \( \alpha \) has the form \( x(x - a) \) for some \( a \in K \). [8 points]

6. Let \( K = \mathbb{R} \) and let \( V \) be a space with inner product \( ( \cdot | \cdot ) \). If \( \alpha \neq 0 \) and \( (\alpha(v)|w) = -(v|\alpha(w)) \) for all \( v, w \in V \), then prove the following:
   (1) There exists an invariant subspace \( W \) of \( V \), with orthonormal basis \( e_1, e_2 \), such that \( \alpha(e_1) = -e_2 \) and \( \alpha(e_2) = e_1 \). [8 points]
   (2) The orthogonal complement \( W^\perp \) of \( W \) is \( \alpha \)-invariant. [6 points]
   (3) There exists an orthonormal basis of \( V \) in which the matrix of \( \alpha \) has the form
\[
\begin{bmatrix}
  A_1 & 0 & \cdots & 0 & 0 \\
  0 & A_2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & A_k & 0 \\
  0 & 0 & \cdots & 0 & O_{n-2k}
\end{bmatrix}
\]
where \( A_i = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix} \) with \( a_i \in K \) and \( O_{n-2k} \) is the zero matrix of order \( n - 2k \). [6 points]

7. Let \( \beta : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3 \) be a homomorphism of abelian groups, given by left multiplication with the matrix
\[
\begin{bmatrix}
  -1 & 3 & 2 \\
  0 & 2 & 4 \\
  2 & -2 & 4
\end{bmatrix}.
\]
   (1) Explain why \( \text{Ker } \beta \) is a free abelian group, and find a basis. [8 points]
   (2) Decompose \( \mathbb{Z}^3 / \text{Im } \beta \) as a direct sum of cyclic groups. [8 points]

8. Let \( p \) be a prime number, and \( A = \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^3) \). Compute:
   (1) The number of elements of \( A \) of order \( p^2 \). [8 points]
   (2) The number of cyclic subgroups of \( A \) of order \( p^2 \). [8 points]
   (3) The number of non-cyclic subgroups of \( A \) of order \( p^2 \). [8 points]