In answering any part of a question you may assume the preceding parts.

**Notation:** \( V \) is a finite dimensional vector space over a field \( K \);
\( \alpha: V \to V \) is a linear operator.

1. In some basis of \( V \), \( \alpha \) is given by the matrix
\[
\begin{bmatrix}
1 & 1 & -2 & 0 \\
2 & 1 & 0 & 2 \\
1 & 0 & 1 & 1 \\
0 & -1 & 2 & 1
\end{bmatrix}
\]. Find:

(1) the rational normal form of \( \alpha \). [8]

(2) the Jordan normal form of \( \alpha \). [7]

2. Let \( P \) be the space of polynomials of degree \( < n \) over \( K \), and let \( \delta: P \to P \) be the operator, given by differentiation:
\[
\delta \left( \sum_{i=0}^{n-1} a_i x^i \right) = \sum_{i=1}^{n-1} i a_i x^{i-1}.
\]
Find the Jordan normal form of the \( \delta^2 \), when

(1) \( K \) is the field \( \mathbb{C} \) of complex numbers. [5]

(2) \( K \) is the field \( \mathbb{F}_3 \) with 3 elements. [5]

3. A \( \alpha \)-invariant subspace \( W \leq V \) is called irreducible, if the only proper \( \alpha \)-invariant subspaces of \( W \) are 0 and \( W \) itself.

(1) Prove that if the characteristic polynomial of \( \alpha \) has an irreducible factor of degree \( d \), then \( \alpha \) has an irreducible invariant subspace of dimension \( d \). [10]

(2) Prove the converse of (1). [10]

4. Let \( v_1, v_2 \) and \( w_1, w_2 \) be two pairs of vectors in a real inner product space \( V \).

(1) Prove that if \( \|v_1\| = \|w_1\|, \|v_2\| = \|w_2\|, \) and \( \langle v_1, v_2 \rangle = \langle w_1, w_2 \rangle \), then there is an orthogonal operator \( \alpha: V \to V \), such that \( \alpha(v_1) = w_1 \) and \( \alpha(v_2) = w_2 \). [10]

(2) Give necessary and sufficient conditions for \( \alpha \) in (1) to be unique. [5]

(3) Does the converse of (1) hold? [5]

5. Let \( B \) be the subgroup of \( \mathbb{Z}^3 \) generated by \( (3, 6, 3), (-1, 4, 0), \) and \( (5, 4, 6) \), and let \( A = \mathbb{Z}^3 / B \).

(1) Express \( A \) as a direct sum of cyclic groups. [8]

(2) How many distinct subgroups of order 6 does \( A \) contain? [7]

6. Let \( A \) be a finite abelian group, let \( n \) be an integer, and let \( \beta: A \to A \) be the map, defined by \( \beta(a) = na \) for each \( a \in A \).

(1) Prove that the abelian groups \( \text{Ker}(\beta) \) and \( A / \text{Im}(\beta) \) are isomorphic. [10]

(2) Prove that for each prime number \( p \), the number of subgroups of \( A \) of order \( p \) is equal to the number of subgroups of \( A \) of index \( p \). [Hint: Use the preceding problem with \( n = p \).] [10]