Qualifying Examination **Name:**
Math 554 **January, 1998**

ANSWERING ANY QUESTION YOU CAN USE THE ANSWERS TO PRECEDING ONES

NOTATION. $M_n(F)$ is the set of $n \times n$ **matrices** with elements in a **field** F.

1. For the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \in M_4(\mathbb{R})$ find:

a) The rational form R and Jordan canonical form J. [10]
b) An invertible matrix $S \in M_4(\mathbb{R})$ such that $S^{-1}AS = J$. [5]

2. For $A = (a_{ij}) \in M_n(F)$ with $n \geq 3$, let $A^\dagger = (a_{ij}^\dagger) \in M_n(F)$ be the matrix in which $a_{ij}^\dagger$ is the cofactor A_{ij} of a_{ij}. Prove that $A^{\dagger\dagger} = \det(A)^{n-2}A$. [10]

3. For all $A, B \in M_n(F)$, set $\langle A, B \rangle = \mathrm{tr}(AB)$.
1. Prove that $\langle \; , \; \rangle$ is a non-degenerate symmetric bilinear form on $M_n(F)$. [5]

Fix $C \in M_n(F)$ and set $S = \{A \in M_n(F) : AC = CA\}$.

2. Show that $S^\perp = \{BC - CB : B \in M\}$. [10]

NOTATION. T is a **linear operator** on a non-zero **finite dimensional vector space** V over F.

4. The matrix of T in some basis of V is equal to $\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$.

For each property below, determine those λ and μ for which it holds:

1. The $\mathbb{C}[x]$-module associated with T is cyclic. [3]
2. There are only finitely many T-invariant subspaces. [3]
3. For every T-invariant subspace U of V there exists an T-invariant subspace U' of V such that $V = U \oplus U'$. [3]

5 Assume that the minimal polynomial and the characteristic polynomial of T are equal. Show that a linear operator $S \colon V \to V$ commutes with T if and only if $S = p(T)$ for some polynomial $p(x) \in F[x]$. [10]

6. Assume that V has a positive-definite hermitian inner product over $F = \mathbb{C}$. If T satisfies $TT^* = T^*T$, prove that $T^* = p(T)$ for some polynomial $p(x) \in \mathbb{C}[x]$. [10]

7. Let V have a positive-definite inner product over $F = \mathbb{R}$, and the operator T preserves orthogonality, that is, $u \perp v$ implies that $T(u) \perp T(v)$. Prove that $T = \lambda S$ for some orthogonal operator S and some $\lambda \in \mathbb{R}$. [10]

NOTATION. G is an **abelian group**.

8. List (up to isomorphism) all G with $|G| = 72$ and explain why your list is complete. Determine those among them that contain the largest number of subgroups of order 6. [11]

9. Suppose G and H are abelian groups of finite order having the same number of elements of order n for every positive integer n. Show that G and H are isomorphic.
(Hint : Begin by considering elements of prime order.) [10]