1. For the matrix
\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
-2 & -2 & 2 & 1 \\
1 & 1 & -1 & 0
\end{pmatrix} \in M_4(\mathbb{R})
\]
find:

a) The rational form \( R \) and Jordan canonical form \( J \). \[10\]
b) An invertible matrix \( S \in M_4(\mathbb{R}) \) such that \( S^{-1}AS = J \). \[5\]

2. For \( A = (a_{ij}) \in M_n(F) \) with \( n \geq 3 \), let \( A^\dagger = (a^\dagger_{ij}) \in M_n(F) \) be the matrix in which \( a^\dagger_{ij} \) is the cofactor \( A_{ij} \) of \( a_{ij} \). Prove that \( A^\dagger = \det(A)^{n-2}A \). \[10\]

3. For all \( A, B \in M_n(F) \), set \( (A, B) = \text{tr}(AB) \).

1. Prove that \( (\cdot, \cdot) \) is a non-degenerate symmetric bilinear form on \( M_n(F) \). \[5\]

Fix \( C \in M_n(F) \) and set \( S = \{ A \in M_n(F) : AC = CA \} \).

2. Show that \( S^\perp = \{ BC - CB : B \in M \} \). \[10\]

4. The matrix of \( T \) in some basis of \( V \) is equal to
\[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
1 & \lambda & 0 & 0 \\
0 & 1 & \lambda & 0 \\
0 & 0 & 0 & \mu
\end{pmatrix}
\]

For each property below, determine those \( \lambda \) and \( \mu \) for which it holds:

1. The \( \mathbb{C}[x] \)-module associated with \( T \) is cyclic. \[3\]
2. There are only finitely many \( T \)-invariant subspaces. \[3\]
3. For every \( T \)-invariant subspace \( U \) of \( V \) there exists an \( T \)-invariant subspace \( U' \) of \( V \) such that \( V = U \oplus U' \). \[3\]

5. Assume that the minimal polynomial and the characteristic polynomial of \( T \) are equal. Show that a linear operator \( S : V \to V \) commutes with \( T \) if and only if \( S = p(T) \) for some polynomial \( p(x) \in \mathbb{F}[x] \). \[10\]

6. Assume that \( V \) has a positive-definite hermitian inner product over \( F = \mathbb{C} \). If \( T \) satisfies \( TT^* = T^*T \), prove that \( T^* = p(T) \) for some polynomial \( p(x) \in \mathbb{C}[x] \). \[10\]

7. Let \( V \) have a positive-definite inner product over \( F = \mathbb{R} \), and the operator \( T \) preserves orthogonality, that is, \( u \perp v \) implies that \( T(u) \perp T(v) \). Prove that \( T = \lambda S \) for some orthogonal operator \( S \) and some \( \lambda \in \mathbb{R} \). \[10\]

8. List (up to isomorphism) all \( G \) with \( |G| = 72 \) and explain why your list is complete. Determine those among them that contain the largest number of subgroups of order 6. \[11\]

9. Suppose \( G \) and \( H \) are abelian groups of finite order having the same number of elements of order \( n \) for every positive integer \( n \). Show that \( G \) and \( H \) are isomorphic. (Hint: Begin by considering elements of prime order.) \[10\]