1. Let $A \in \mathbb{C}^{4 \times 4}$ be a diagonal matrix with main diagonal entries 1, 2, 3, 4. Define $T_A : \mathbb{C}^{4 \times 4} \to \mathbb{C}^{4 \times 4}$ by $T_A(B) = AB - BA$.

(i) What is $\text{dim}(\ker(T_A))$?
(ii) What is $\text{dim}(\text{im}(T_A))$?
(iii) What are the eigenvalues of $T_A$?
(iv) What is the minimal polynomial of $T_A$?
(v) Is $T_A$ diagonalizable? Explain.

2. (i) Let $A \in \mathbb{Z}^{3 \times 4}$ and define $\phi_A : \mathbb{Z}^4 \to \mathbb{Z}^3$ by $\phi_A(X) = AX$.

True or False? If $\phi_A$ is surjective, then the determinant of some $3 \times 3$ minor of $A$ is a unit of $\mathbb{Z}$. Explain.

(ii) Let $B \in \mathbb{Z}^{4 \times 3}$ and define $\phi_B : \mathbb{Z}^3 \to \mathbb{Z}^4$ by $\phi_B(X) = BX$.

True or False? If the determinant of some $3 \times 3$ minor of $B$ is nonzero, then $\phi_B$ is injective. Explain.

3. True or False? If $A \in \mathbb{R}^{n \times n}$ is normal and if the eigenvalues of $A$ are all real, then $A$ is symmetric. Justify your answer.

4. Let $V$ be a vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

5. Let $V$ be a vector space over an infinite field $F$. Suppose $\alpha_1, \ldots, \alpha_m$ are finitely many nonzero vectors in $V$. Prove there exists a linear functional $f$ on $V$ such that $f(\alpha_i) \neq 0$ for each $i$.

6. Let $V$ be an abelian group generated by $a, b, c$, where $2a = 4b, 2b = 4c, 2c = 4a$, and where these 3 relations generate all the relations on $a, b, c$.

(i) For some positive integer $n$, find elements $x_1, \ldots, x_n \in V$ that generate $V$ and have the property that $c_i \in \mathbb{Z}$ with $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ implies each $c_ix_i = 0$.

(ii) Write $V$ as a direct sum of cyclic groups. What is the order of $V$?

7. Let $F$ be a field, let $m$ and $n$ be positive integers, and let $F^{m \times n}$ denote the
set of $m \times n$ matrices with entries in $F$.

(i) What does it mean for $R \in F^{m \times n}$ to be a row-reduced echelon matrix?

(ii) Suppose $W$ is a subspace of $F^n$ with $\dim W \leq m$. Prove there is precisely one row-reduced echelon matrix $R \in F^{m \times n}$ such that $W$ is the row space of $R$.

(12) 8. Suppose $F$ is a field of characteristic zero and $V$ is a finite-dimensional vector space over $F$. If $E_1, \ldots, E_k$ are projection operators of $V$ such that $E_1 + \cdots + E_k = I$, the identity operator on $V$, prove that $E_i E_j = 0$ for $i \neq j$.

(10) 9. Prove or disprove: if $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear operator such that every subspace of $\mathbb{R}^4$ is invariant under $T$, then $T$ is a scalar multiple of the identity operator.

(18) 10. Suppose $\mathcal{F}$ is a subspace of $\mathbb{C}^{4 \times 4}$ such that for each $A, B \in \mathcal{F}$, $AB = BA$. If there exists $A \in \mathcal{F}$ having at least two distinct characteristic values, prove that $\dim \mathcal{F} \leq 4$.

(22) 11. Assume that $V$ is a finite-dimensional vector space over an infinite field $F$ and $T : V \to V$ is a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x \alpha = T(\alpha)$ for each $\alpha \in V$.

(1) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.

(2) In terms of the expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant submodules? Explain.

(18) 12. Assume that $M$ is a module over an integral domain $D$. Recall that a submodule $N$ of $M$ is said to be pure if for each $y \in N$ and $a \in D$, $ax = y$ is solvable in $M$ if and only if it is solvable in $N$.

(1) If $N$ is a direct summand of $M$, prove that $N$ is pure in $M$.

(2) For $x \in M$, let $x + N$ denote the coset representing the image of $x$ in the quotient module $M/N$. If $N$ is a pure submodule of $M$ and $\text{ann}(x + N)$ is a principal ideal $(d)$ of $D$, prove that there exists $x' \in D$ such that $x + N = x' + N$ and $\text{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal $(d)$. 
Assume that $M$ is a finitely generated torsion module over the polynomial ring $F[x]$, where $F$ is a field, and that $N$ is a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$. 