(7) 1. Let $V$ be an abelian group and assume that $(v_1, \ldots, v_m)$ are generators of $V$. Describe a process for obtaining an $m \times n$ matrix $A \in \mathbb{Z}^{m \times n}$ such that if $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ is the $\mathbb{Z}$-module homomorphism defined by left multiplication by $A$, then $V \cong \mathbb{Z}^m / \phi(\mathbb{Z}^n)$. Such a matrix $A$ is called a presentation matrix of $V$.

(15) 2. Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(1) Write down a presentation matrix for $V$ as a $\mathbb{Z}$-module.

(2) Let $W$ be the cyclic subgroup of $V$ generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z} = V$. Write down a presentation matrix for $W$.

(3) Write down a presentation matrix for the quotient module $V/W$.

(20) 3. Let $R$ be a commutative ring and let $V$ and $W$ denote free $R$-modules of rank 4 and 5, respectively. Assume that $\phi : V \to W$ is an $R$-module homomorphism, and that $B = (v_1, \ldots, v_4)$ is an ordered basis of $V$ and $B' = (w_1, \ldots, w_5)$ is an ordered basis of $W$.

(1) What is meant by the coordinate vector of $v \in V$ with respect to the basis $B$?

(2) Describe how to obtain a matrix $A \in R^{5 \times 4}$ so that left multiplication by $A$ on $R^4$ represents $\phi : V \to W$ with respect to $B$ and $B'$.

(3) How does the matrix $A$ change if we change the basis $B$ by replacing $v_1$ by $v_1 + v_2$?

(4) How does the matrix $A$ change if we change the basis $B'$ by replacing $w_1$ by $w_1 + w_2$?

(18) 4. Let $A$ be an $4 \times 5$ matrix with integer coefficients and let $\phi : \mathbb{Z}^5 \to \mathbb{Z}^4$ be defined by left multiplication by $A$.

(1) Prove or disprove: if $\phi$ is surjective, then the determinants of the $4 \times 4$ minors of $A$ generate the unit ideal of $\mathbb{Z}$.

(2) Prove or disprove: if $\phi$ is surjective, then there exists a matrix $B \in \mathbb{Z}^{5 \times 4}$ such that $AB$ is the $4 \times 4$ identity matrix.

(10) 5. Let $V = \mathbb{Z}^2$ and let $L$ be the submodule of $V$ spanned by the columns of $A = \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$. Find a basis $(\alpha_1, \alpha_2)$ of $V$ and integers $c_1, c_2$ so that $c_1\alpha_1, c_2\alpha_2$ is a basis for $L$.

(10) 6. If $A \in \mathbb{R}^{n \times n}$ is symmetric, does there exist $B \in \mathbb{R}^{n \times n}$ such that $B^3 = A$? Justify your answer.

(16) 7. Let $F$ be a field and let $F[t]$ be a polynomial ring in one variable over $F$. Let $r$ and $s$ and $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_s$ be positive integers. Suppose

$$V = F[t]/(t^{a_1}) \oplus F[t]/(t^{a_2}) \oplus \cdots \oplus F[t]/(t^{a_r})$$
and
\[ W = F[t]/(t^{b_1}) \oplus F[t]/(t^{b_2}) \oplus \cdots \oplus F[t]/(t^{b_s}). \]

If the $F[t]$-modules $V$ and $W$ are isomorphic, prove the structure theorem that asserts that $r = s$, and that $a_i = b_i$ for $i = 1, \ldots, r$.

(10) 8. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and let $f(x) \in \mathbb{C}[x]$ be a monic polynomial. Suppose $a \in \mathbb{C}$ is an eigenvalue of $f(T)$. Prove or disprove that there must exist an eigenvalue $b$ of $T$ such that $f(b) = a$.

(10) 9. Suppose $A \in \mathbb{R}^{5 \times 3}$ has rank 3. Let $A^T$ denote the transpose of $A$. Prove or disprove that $A^TA \in \mathbb{R}^{3 \times 3}$ must be nonsingular.

(12) 10. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a linear operator that preserves orthogonality, i.e., if $u \perp v$, then $T(u) \perp T(v)$. Prove that $T = \lambda S$ for some orthogonal operator $S$ and some $\lambda \in \mathbb{R}$.

(10) 11. Suppose $A \in \mathbb{R}^{n \times n}$. If $A$ is normal and if the eigenvalues of $A$ are all real, does it follow that $A$ is symmetric? Justify your answer with a proof or a counterexample.

(10) 12. If $v$ is a nonzero vector in $\mathbb{R}^3$ and $w$ is a nonzero vector in $\mathbb{R}^5$, must there exist a 3 by 5 matrix $A$ whose column space is spanned by $v$ and whose row space is spanned by $w$? Justify your answer.

(10) 13. Let $V$ be a vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

(10) 14. Suppose $S : \mathbb{C}^5 \to \mathbb{C}^5$ and $T : \mathbb{C}^5 \to \mathbb{C}^5$ are commuting linear operators. Prove that there exists a nonzero $v \in \mathbb{C}^5$ which is an eigenvector for both $S$ and $T$.

(10) 15. If $A$ and $B$ in $\mathbb{R}^{n \times n}$ are normal matrices, does it follow that $AB$ is also normal? Justify your answer.

(10) 16. Is $A \in \mathbb{C}^{n \times n}$ always similar to its transpose $A^T$? Justify your answer.

(12) 17. Classify up to similarity all matrices $A \in \mathbb{C}^{3 \times 3}$ such that $A^3 = I$, i.e., write down all possibilities for the Jordan form of $A$. 