QUALIFYING EXAMINATION

JANUARY 2000

MATH 554 - Prof. Wang

Each problem is worth 10 points.

1. Let $AX = B$ and $A_1X = B_1$ be two consistent systems of linear equations. If they have the same set of solutions, prove that they are equivalent.
2. Let V be an n-dimensional subspace of $\mathbb{Q}[X]$ over $\mathbb{Q}$. Prove that there exist $f_1, \ldots, f_n \in V$ and positive integers $m_1, \ldots, m_n$ such that $f_i(m_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.
3. Let A be a linear operator on a finite dimensional vector space V over a field F. Show that

$$\text{rank}(A^2) + \text{rank}(A^7) \geq \text{rank}(A^5) + \text{rank}(A^4).$$

4. Let F be a field and $A, B \in M_{nn}(F)$. Show that AB and BA have the same characteristic polynomial.
5. Let V be a finite dimensional vector space over a field of characteristic 0, $A \in L(V, V)$ and T_A the linear operator on $L(V, V)$ given by $T_A(B) = AB - BA$. Assume that B is a characteristic vector of T_A with nonzero characteristic value. Show that B is nilpotent.
6. Let V be a finite dimensional vector space over a field F with $|F| > 2$ and $A \in L(V, V)$. Show that there exist $B, C \in L(V, V)$ such that
   (i) $A = B + C$,
   (ii) both B and C have cyclic vectors.
7. Let $A \in M_{6\times 6}(\mathbb{Q})$ satisfying $A^3 = I$. Write out the possible rational forms for A.
8. Let $A \in M_{nn}(\mathbb{R})$ satisfying $A^tA = AA^t$. Show that there exists a real polynomial $f(X)$ such that $f(A) = A^t$.
9. Let $A, B \in M_{nn}(\mathbb{C})$. Assume that $A^* = A$, $B^* = B$, $\text{tr}(A) = \text{tr}(B)$ and $X^*AX \geq X^*BX$ for all $X \in M_{n\times 1}(\mathbb{C})$. Show that $A = B$.
10. Let F be a field of characteristic 2. Give an example of a vector space V over F and distinct projections E_1, E_2, E_3 of V such that
    (i) $E_1 + E_2 + E_3 = I$
    (ii) $E_iE_j \neq 0$ for $i \neq j$.