1. (30 points) Determine whether the following statements are true or false (you must justify the answer with either a proof or a counterexample). All matrices and vector spaces in this problem are defined over the field $\mathbb{R}$.

   a) Let $A$ and $B$ be $n \times n$ matrices. Then $AB$ is invertible if and only if $A$ and $B$ are.

   b) Every square matrix is a product of elementary matrices.

   c) There exists a $3 \times 2$ matrix $A$ and a $2 \times 3$ matrix $B$ such that $AB$ is invertible.

   d) If $V$ and $W$ are finite dimensional vector spaces such that $\dim V \leq \dim W$, then $V$ is isomorphic to a subspace of $W$.

   e) If $v_1, v_2, v_3$ are three distinct nonzero vectors in a finite dimensional vector space $V$, there exists a linear transformation $f : V \to \mathbb{R}$ satisfying $f(v_i) = i$ for all $i$.

   f) If $v_1, v_2, v_3$ are three linearly independent vectors in a finite dimensional vector space $V$, there exists a linear transformation $f : V \to \mathbb{R}$ satisfying $f(v_i) = i$ for all $i$.

2. (10 points) Let $V$ be a finite dimensional vector space. Suppose that $W_1, W_2 \subset V$ are subspaces. Define the linear transformation $L : W_1 \oplus W_2 \to V$ by $L(w_1, w_2) = w_1 + w_2$. Calculate the kernel and image of $L$, and use this to prove that

   $$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

3. (20 points) Let $A$ and $B$ be $n \times n$ matrices over a field $F$. Suppose that $A$ is invertible.

   a) If $F$ is infinite, prove that there exists $\lambda \in F$ such that $\lambda A + B$ is invertible.

   b) Give an example to show that the conclusion of part a) can fail when $F = \mathbb{Z}/2\mathbb{Z}$.

4. (20 points) Let $A$ be an $n \times n$ matrix with entries in a field $F$. Assume that $A$ is idempotent i.e. $A^2 = A$. Let $L : F^n \to F^n$ be the corresponding linear transformation defined by $Lv = Av$.

   a) Prove that the only possible eigenvalues for $A$ are 0 and 1.

   b) Prove that $v$ is an eigenvector of $A$ with eigenvalue 1 if and only if $v$ lies in the image $\text{im}(L)$.

   c) Prove that $F^n = \text{ker}(L) + \text{im}(L)$ and that $\text{ker}(L) \cap \text{im}(L) = 0$. 


d) Let \( B = (b_{ij}) \) be the matrix representing \( L \) in a given basis \( v_1, \ldots, v_n \) of \( F^n \), i.e. \( Lv_i = \sum_j b_{ij}v_j \). Show that the basis can be chosen so that
\[
B = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
0 & \cdots & 1
\end{pmatrix}
\]

5. (10 points) Let \( S \subset \mathbb{Z}^3 \) be the sub-abelian group generated by \((2, 2, 2)^T\) and \((3, 1, 1)^T\). Express \( \mathbb{Z}^3/S \) as a direct sum of cyclic groups.

6. (10 points) Let
\[
A = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & -3 \\
0 & 1 & 3
\end{pmatrix}
\]
and let \( M = \mathbb{C}^3 \) with the \( \mathbb{C}[x] \)-module structure determined by the rule \( p(x)v = p(A)v = (a_n A^n + \ldots + a_0 I)v \) for \( p(x) = a_n x^n + \ldots + a_0 \in \mathbb{C}[x], v \in M \). Find a polynomial \( f(x) \) such that \( M \) is isomorphic to \( \mathbb{C}[x]/(f) \). (Hint: Consider the \( \mathbb{C}[x] \)-module homomorphism \( \phi : \mathbb{C}[x] \to M \) which sends 1 to \((1, 0, 0)^T\).)

7. (20 points) Suppose that \( A \) is a \( 2 \times 2 \) matrix over an algebraically closed field \( F \).
   a) Prove that \( A \) is either diagonalizable or similar to a matrix of the form
   \[
   \begin{pmatrix}
   \lambda & 1 \\
   0 & \lambda
   \end{pmatrix}
   \]
   , and show that these possibilities are mutually exclusive.
   b) Prove that \( A^2 \) is always diagonalizable if \( F \) has characteristic 2.