Math 554

1. Let $A \in M_{4\times 5}(\mathbb{Z})$ and define
$$\phi_A : \mathbb{Z}^5 \longrightarrow \mathbb{Z}^4$$
by $\phi_A(X) = AX,\ X \in \mathbb{Z}^5$. Is it true that if ϕ_A is surjective, then the determinant of some 4×4 minor of A is a unit in $\mathbb{Z}$? Justify your answer.

(10 points)

2. Suppose A and B are two commuting linear operators on a finite dimensional complex vector space. Show that they share a non–zero eigenvector.

(10 points)

3. Determine all the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ whose kernel and image are equal. Is this possible for $\mathbb{R}^5$? Justify your answer.

(10 points)

4. A linear transformation N is nilpotent if $N^m = 0$ for some non–negative integer m. Suppose N_1 and N_2 are nilpotent. Is it true that $N_1 N_2$ is nilpotent? Justify your answer.

(10 points)

5. Suppose $A \in M_{5\times 6}(\mathbb{R})$ has rank 5. Prove or disprove that $AA^T \in M_{5\times 5}(\mathbb{R})$ is non–singular.

(15 points)

6. Let $A = \mathbb{R}[x]$ be the polynomial ring in one variable x. Express the A–module $A^3/\langle f_1, f_2, f_3\rangle$, where $f_1 = (x-2, 0, 0)$, $f_2 = (4, 4, -x)$, and $f_3 = (x, x, -1)$, as a direct sum of cyclic A–modules. In particular, state whether this module is a cyclic A–module.

(20 points)

7. Classify all the non–isomorphic finite abelian groups of order 9800.

(10 points)

8. Write the minimal polynomial $m_T(x)$, rational canonical form R, and Jordan Canonical form J for

$$T = \begin{bmatrix} 6 & 3 & 6 \\ 1 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix}.$$

(25 points)

9. a) Define the following notions for a complex inner product space:

   1) adjoint A^* of an operator A,

   2) a Hermitian (self–adjoint) operator,

   3) a skew–Hermitian operator,

   4) a unitary operator,

   5) a normal operator.

   In what follows all the inner product spaces are finite dimensional.

   b) Give an example of an infinite set of normal operators neither of which are Hermitian, nor skew–Hermitian, nor unitary.

   c) Show that an operator A is normal iff $A^* = f(A)$ for some $f(x) \in \mathbb{C}[x]$.

   d) Suppose A is both normal and nilpotent, i.e. $A^m = 0$ for some non–negative integer m. Show that $A = 0$.

   d) Suppose A is normal and $A^6 = A^5$. Conclude that A is idempotent, i.e. $A^2 = A$.

(20 points)

10. Let N be a normal operator on a finite dimensional complex inner product space. Let W be a subspace invariant under N, i.e. $N(W) \subset W$. Show that $W^{\perp}$ is also invariant under N.

(10 points)

11. Determine all the 2×2 complex matrices which are both skew–Hermitian and unitary and conclude that every such matrix is a product of an imaginary scalar matrix and Hermitian matrix whose diagonal entries are either 1 and -1, or zeros. Note that non–unitary skew–Hermitian 2×2 complex matrices give an infinite family of normal operators which are neither unitary nor Hermitian.

(10 points)