QUALIFYING EXAMINATION
JANUARY 2002
MATH 554 - PROF. MATSUKI

NAME ID #

1. Let A be an Abelian group satisfying conditions

(α) $(5^3 \cdot 7^4)A = 0$, and

(β) $\#\{a \in A; (5 \cdot 7)a = 0\} = 5^2 \cdot 7^2$.

(i) Show that A is finitely generated. (10 points)

(ii) How many such Abelian groups that satisfy conditions (α) and (β) do we have up to isomorphism ? (10 points)

2. Let A be the following 3×3 matrix

$$A = \begin{pmatrix} 0 & 2 & 1 \\ -4 & 6 & 2 \\ 4 & -4 & 0 \end{pmatrix}.$$

(i) Find the rational canonical form C of A, and an invertible matrix S such that $C = S^{-1}AS$. (10 points)

(ii) Find the Jordan canonical form J of A, and an invertible matrix T such that $J = T^{-1}AT$. (10 points)

3. Let A be a 3×3 matrix whose Jordan canonical form is

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix}.$$

Let B be a 4×4 matrix whose Jordan canonical form is

$$\begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Consider the space V of 3×4 matrices X with entries in $\mathbb{C}$ such that

$$AX = XB.$$

Find the dimension of the vector space V over $\mathbb{C}$. (20 points)

4. Let $V = \mathbb{C}^4$ be a 4-dimensional vector space over $\mathbb{C}$ and let $f : V \to V$ be a linear map defined by $f(v) = Tv$ where

$$T = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

(Note that an element $v \in \mathbb{C}^4$ is considered to be a column vector.)

(i) Find the criterion on α and β so that V has only a finite number of f-invariant subspaces if and only if this criterion is satisfied. (10 points)

(ii) Find the number of f-invariant subspaces of dimension 2 when α and β satisfy the above criterion. (10 points)

Note: A subspace $W \subset V$ is said to be f-invariant for f if $f(W) \subset W$.

5. How many 2×2 matrices A with entries complex numbers are there, up to similarity, with property that the minimal polynomial of A^2 coincides with that of A ? (20 points)

6. Let A be an $n \times n$ complex matrix. Show that

$$|\det(A)| \leq ||a_1||||a_2|| \cdots ||a_n||,$$

where a_i denotes the i-th column vector of A and where

$$||a_i|| = \sqrt{a_{1i} \cdot \overline{a_{1i}} + \cdots + a_{ni} \cdot \overline{a_{ni}}}.$$

(20 points)

7. Let $A_1, ..., A_m$ be $n \times n$ Hermitian matrices. Show that $\Sigma_{i=1}^m A_i^2 = 0$ if and only if $A_i = 0$ for $i = 1, ..., m$. (20 points)

8. How many 3×3 (real) orthogonal matrices are there, up to similarity, with property $\det(A) = -1$ and $A^3 = A$? (20 points)

9. To a quadratic form over $\mathbb{R}$ in n variables

$$Q(x_1, ..., x_n) = \Sigma a_{ij} x_i x_j \quad \text{with } a_{ij} = a_{ji} \text{ and } a_{ij} \in \mathbb{R},$$

we associate a real $n \times n$ symmetric matrix $A = (a_{ij})$.

Consider a 3×3 matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

and regard

$$Q(x_{11}, ..., x_{33}) = \text{Trace}(X^2)$$

as a real quadratic form in 9 variables. Associate the corresponding real 9×9 matrix A to Q as above.

Find the numbers of positive, negative, and zero eigenvalues of A, respectively.

(20 points)

10. Let H be an $n \times n$ Hermitian matrix with only positive eigenvalues. Show that, for any complex $n \times n$ matrix A, the matrices H and A commute if and only if H^2 and A commute, i.e.,

$$AH = HA \Longleftrightarrow H^2 A = AH^2.$$

(20 points)