1. Let $F$ be a field, let $n$ be a positive integer, and let $W = F^{n \times n}$ denote the vector space of $n \times n$ matrices with entries in $F$.

(i) Let $W_0$ denote the subspace of $W$ spanned by the matrices $C$ of the form $C = AB - BA$. What is $\dim W_0$?

(ii) For $A \in F^{n \times n}$, define the adjoint matrix $\text{adj} A \in F^{n \times n}$.

(iii) If $A \in \mathbb{R}^{3 \times 3}$ and $\det A = 2$, what is $\det \text{adj} A$?

2. Let $T : \mathbb{C}^5 \to \mathbb{C}^5$ be a linear operator and let $g(x)$ be a polynomial in $\mathbb{C}[x]$.

If $c$ is a characteristic value for $g(T)$, must there exist a characteristic value $a$ for $T$ such that $g(a) = c$? Explain why or why not.
3. Let $A \in \mathbb{C}^{4 \times 4}$ be a diagonal matrix with main diagonal entries 1, 2, 3, 4. Define $T_A : \mathbb{C}^{4 \times 4} \to \mathbb{C}^{4 \times 4}$ by $T_A(B) = AB - BA$.

(i) What is the dimension of the null space of $T_A$?

(ii) What is the dimension of the range of $T_A$?

(iii) What are the characteristic values of $T_A$?

(iv) What is the minimal polynomial of $T_A$?

(v) Is $T_A$ diagonalizable? Explain.
4. Let $F$ be a field, let $m$ and $n$ be positive integers and let $A \in F^{m \times n}$ be an $m \times n$ matrix.

(i) Define “row space of $A$”.

(ii) Define “column space of $A$”.

(iii) Prove that the dimension of the row space of $A$ is equal to the dimension of the column space of $A$. 
5. Let $D$ be a principal ideal domain and let $V$ and $W$ denote free $D$-modules of rank 4 and 5, respectively. Assume that $\phi : V \to W$ is a $D$-module homomorphism, and that $B = \{v_1, \ldots, v_4\}$ is an ordered basis of $V$ and $B' = \{w_1, \ldots, w_5\}$ is an ordered basis of $W$.

(i) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis $B$?

(ii) Describe how to obtain a matrix $A \in D^{5 \times 4}$ so that left multiplication by $A$ on $D^4$ represents $\phi : V \to W$ with respect to $B$ and $B'$.

(iii) How does the matrix $A$ change if we change the basis $B$ by replacing $v_1$ by $v_1 + av_2$ for some $a \in D$?

(iv) How does the matrix $A$ change if we change the basis $B'$ by replacing $w_1$ by $w_1 + aw_2$ for some $a \in D$?
(10) 6. Classify up to similarity all matrices $A \in \mathbb{C}^{3 \times 3}$ such that $A^3 = I$, where $I$ is the identity matrix, i.e., write down all possibilities for the Jordan form of $A$.

(10) 7. List up to isomorphism all abelian groups of order 72.
8. Let $V$ be a finite-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator. Give $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(i) If $\{v_1, \cdots, v_n\}$ are generators for $V$ as an $F[x]$-module, what does it mean for $A \in F[x]^{m \times n}$ to be a relation matrix for $V$ with respect to $\{v_1, \ldots, v_n\}$?

(ii) If $F = \mathbb{C}$ and $A = \begin{bmatrix}
  x^2(x - 1)^2 & 0 & 0 \\
  0 & x(x - 1)(x - 2)^2 & 0 \\
  0 & 0 & x(x - 2)^3
\end{bmatrix}$ is a relation matrix for $V$ with respect to $\{v_1, v_2, v_3\}$, list the invariant factors of $V$.

(iii) With assumptions as in part (ii), list the elementary divisors of $V$ and describe the direct sum decomposition of $V$ given by the primary decomposition theorem.

(iv) With assumptions as in part (ii), write the Jordan form of the operator $T$. 
9. Let $V$ be a finite-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator such that $\text{rank } T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of $T$. Explain.

10. Let $V$ be an abelian group with generators $\{v_1, v_2, v_3\}$ that has the matrix

\[
\begin{pmatrix}
2 & 0 & 6 \\
4 & 8 & 0
\end{pmatrix}
\]

as a relation matrix. Express $V$ as a direct sum of cyclic groups.
11. Let $V$ be an abelian group generated by elements $a, b, c$. Assume that $2a = 6b, 2b = 6c, 2c = 6a$, and that these three relations generate all the relations on $a, b, c$.

(i) Write down a relation matrix for $V$.

(ii) Find generators $x, y, z$ for $V$ such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by $x, y, z$. Express your generators $x, y, z$ in terms of $a, b, c$. What is the order of $V$?
(8) 12. Let $F$ be a field.

(i) What is the dimension of the vector space of all 3-linear functions $D : F^{3 \times 3} \to F$? Explain why.

(ii) What is the dimension of the vector space of all 3-linear alternating functions $D : F^{3 \times 3} \to F$? Explain why.

(10) 13. Prove or disprove: if $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear operator that has a cyclic vector and $S : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear operator that commutes with $T$, then $S$ is a polynomial in $T$. 
14. Assume that $V$ is a finite-dimensional vector space over an infinite field $F$ and $T : V \to V$ is a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(i) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.

(ii) In terms of the expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant submodules? Explain.
(14) 15. Assume that $M$ is a module over an integral domain $D$. Recall that a submodule $N$ of $M$ is said to be pure in $M$ if for any $y \in N$ and $a \in D$, if there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(i) If $N$ is a direct summand of $M$, prove that $N$ is pure in $M$

(ii) For $x \in M$, let $x + N$ denote the coset representing the image of $x$ in the quotient module $M/N$. If $N$ is a pure submodule of $M$ and $\text{ann}(x + N)$ is a principal ideal $(d)$ of $D$, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\text{ann} x' = \{a \in D : ax' = 0\}$ is the principal ideal $(d)$. 
Assume that $M$ is a finitely generated torsion module over the polynomial ring $F[x]$, where $F$ is a field, and that $N$ is a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$. 