Qualifying Examination
August 2007
Math 554 – Professor Wang

(Choose eight problems. Each: 10 points, total 80 points)

1. Let V be the span of all projections E in $M_{nn}(F)$ with $\text{rank}(E) = 2$ and $n \geq 3$. Assume that F is a field of characteristic 2. Compute the dimension of V.

2. Let V be a finite dimensional vector space over an uncountable field F and V_i, $i = 1, \ldots, n, \ldots$ a sequence of subspaces of V. Show that if $V = U_{i=1}^{\infty} V_i$, then one of V_i is V.

3. Let ℓ be a positive integer and p a prime number, $A \in M_{nn}(\mathbb{Q})$ and $f(x) = x^\ell - px^{\ell-1} - px + p$. Assume that $f(A) = 0$ and $tr(A) = mp$. Compute $tr(A^{-1})$.

4. Let $a_{ij} = i^2 + ij + j^2$ and $f_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $i = 1, \ldots, n$. Determine the dimension of the span of f_i.

5. Let A, B and C be $n \times n$ matrices over a commutative ring R with identity. If $AB = BA$, then
$$\det(A - BC) = \det(A - CB).$$

6. Let A be a nilpotent $n \times n$ matrix and $f(x) = a_0 + a_1x + \cdots + a_\ell x^\ell$ with $a_1 \neq 0$. Show that $f(A)$ and $a_0I + A$ are similar.

7. Let A be an operator of a finite dimensional vector space over $\mathbb{C}$. Assume that A has minimal polynomial $x^2(x + 1)^2$ and characteristic polynomial $x^4(x + 1)^4$. Determine all possible similar classes of A.

8. Let V be an n–dimensional subspace of the space of polynomials over $\mathbb{C}$. Show that there exist $f_1, \ldots, f_n \in V$ and an integer ℓ such that
$$f_i(\ell + j) = \delta_{ij}\ , \ i, j = 1, \ldots, n.$$

9. Let $A_{ij} \in M_{\ell\times\ell}(F) \quad ,i,j = 1,\dots,m$ be matrices and $A = [A_{ij}]$ the $n \times n$ matrix with $n = \ell m$ and the given partition. Assume that A is invertible and A_{ij} commute one another. Show that if $A^{-1} = [B_{ij}]$ with same partition, then B_{ij} commute one another.

10. Let $A_1, \dots, A_k \in M_{nn}(\mathbb{C})$ such that

$$A_i^2 = A_i (i = 1, \dots, k) \quad \text{and} \quad A_1 + \cdots + A_k = I.$$

For $a_1, \dots a_n \in \mathbb{C}$, give a simple expression for $\det\left(\sum_{i=1}^{k} a_i A_i\right)$.

11 Let $A \in M_{nn}(F)$ with $A^T = -A$. If characteristic of $F \neq 2$ and F is infinite, there exist $X, Y \in M_{nn}(F)$ such that

$$X^T = X, \quad Y^T = Y \quad \text{and} \quad A = [X, Y] = XY - YX.$$

12 Let N_1, N_2 be nilpotent operators of a finite dimensional vector space. Assume that N_1 is cyclic and $N_1N_2 = N_2N_1$. Show that N_1 and $N_1 + N_1N_2$ are similar.