(Choose eight problems. Each: 10 points, total 80 points)

1. Let $V$ be the span of all projections $E$ in $M_{nn}(F)$ with rank($E$) = 2 and $n \geq 3$. Assume that $F$ is a field of characteristic 2. Compute the dimension of $V$.

2. Let $V$ be a finite dimensional vector space over an uncountable field $F$ and $V_i$, $i = 1, \ldots, n, \ldots$ a sequence of subspaces of $V$. Show that if $V = \bigcup_{i=1}^{\infty} V_i$, then one of $V_i$ is $V$.

3. Let $\ell$ be a positive integer and $p$ a prime number, $A \in M_{nn}(\mathbb{Q})$ and $f(x) = x^\ell - px^{\ell-1} - px + p$. Assume that $f(A) = 0$ and $tr(A) = mp$. Compute $tr(A^{-1})$.

4. Let $a_{ij} = i^2 + ij + j^2$ and $f_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $i = 1, \ldots, n$. Determine the dimension of the span of $f_i$.

5. Let $A, B$ and $C$ be $n \times n$ matrices over a commutative ring $R$ with identity. If $AB = BA$, then  
$$
\det(A - BC) = \det(A - CB).
$$

6. Let $A$ be a nilpotent $n \times n$ matrix and $f(x) = a_0 + a_1x + \cdots + a_\ell x^\ell$ with $a_1 \neq 0$. Show that $f(A)$ and $a_0I + A$ are similar.

7. Let $A$ be an operator of a finite dimensional vector space over $\mathbb{C}$. Assume that $A$ has minimal polynomial $x^2(x + 1)^2$ and characteristic polynomial $x^4(x + 1)^4$. Determine all possible similar classes of $A$.

8. Let $V$ be an $n$–dimensional subspace of the space of polynomials over $\mathbb{C}$. Show that there exist $f_1, \ldots, f_n \in V$ and an integer $\ell$ such that 
$$
f_i(\ell + j) = \delta_{ij}, \ i, j = 1, \ldots, n.
$$
9. Let $A_{ij} \in M_{\ell \times \ell}(F)$, $i, j = 1, \ldots, m$ be matrices and $A = [A_{ij}]$ the $n \times n$ matrix with $n = \ell m$ and the given partition. Assume that $A$ is invertible and $A_{ij}$ commute one another. Show that if $A^{-1} = [B_{ij}]$ with same partition, then $B_{ij}$ commute one another.

10. Let $A_1, \ldots, A_k \in M_{nn}(\mathbb{C})$ such that

$$A_i^2 = A_i (i = 1, \ldots, k) \text{ and } A_1 + \cdots + A_k = I.$$

For $a_1, \ldots, a_n \in \mathbb{C}$, give a simple expression for $\det\left( \sum_{i=1}^k a_i A_i \right)$.

11. Let $A \in M_{nn}(F)$ with $A^T = -A$. If characteristic of $F \neq 2$ and $F$ is infinite, there exist $X, Y \in M_{nn}(F)$ such that

$$X^T = X, \quad Y^T = Y \quad \text{and} \quad A = [X, Y] = XY - YX.$$

12. Let $N_1, N_2$ be nilpotent operators of a finite dimensional vector space. Assume that $N_1$ is cyclic and $N_1 N_2 = N_2 N_1$. Show that $N_1$ and $N_1 + N_1 N_2$ are similar.