Instructions: Give a complete solution to each question. For problems with multiple parts you may assume the result of the previous parts to solve the subsequent parts. Begin each problem on a new sheet of paper. Be sure your name is on every sheet of your solutions.

Notation: The following are standard for this examination. If \( R \) is a ring, \( M_n(R) \) is the collection of \( n \times n \) matrices with \( R \)-entries, and \( R[x] \) is the ring of polynomials with \( R \)-coefficients. The symbols \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) denote the integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. The symbol \( I_n \) denotes the \( n \times n \) identity matrix, and \( I_V \) is the identity transformation of a vector space \( V \).

1. (10 points) Let \( R \) be a principal ideal domain. A finitely generated \( R \)-module \( M \) is said to be \textbf{indecomposable} if no submodule of \( M \) is a direct summand of \( M \), i.e., it is impossible to find proper submodules \( M_1, M_2 \) of \( M \) so that \( M = M_1 \oplus M_2 \). Determine all indecomposable \( R \)-modules.

2. Let \( R \) be a commutative ring with identity \( 1_R \).
   
   (a) (6 points) Suppose \( A \in M_n(R) \) and \( b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \) is a solution to \( Ax = 0 \).
   
   Show that, for each \( i \), \( b_i \det A = 0 \).
   
   (b) (3 points) Use (a) to show that if \( R \) is an integral domain and \( A \in M_n(R) \) is singular (i.e., the kernel of \( A \) is a non-zero submodule of \( R^n \)) then \( \det A = 0 \).

3. (10 points) Suppose \( A \in M_9(\mathbb{C}) \), and \( I = I_9 \) satisfy the following conditions:
   
   i) \( \text{rank}(A + 2I) = 8 \), and \( \text{rank}(A + 2I)^k = 7 \), for \( k \geq 2 \);
   
   ii) \( \text{rank}(A - (2i)I) = 7 \), and \( \text{rank}(A - (2i)I)^k = 6 \), for \( k \geq 2 \);
   
   iii) \( \text{rank}(A - 3I) = 8 \), \( \text{rank}(A - 3I)^2 = 7 \), \( \text{rank}(A - 3I)^3 = 6 \), and \( \text{rank}(A - 3I)^k = 5 \), for \( k \geq 4 \).

Find the Jordan Canonical form of \( A \).

4. Let \( V \) be a real or complex inner product space, with given inner product \( (\cdot, \cdot) \).
   
   (a) (4 points) Prove that any collection of non-zero orthogonal vectors in \( V \) is linearly independent.
   
   (b) (5 points) Let \( \{v_1, v_2, \ldots, v_n\} \) be an orthogonal subset of \( V \). Prove that, for any \( w \in V \),
   
   \[
   ||w||^2 \geq \sum_{i=1}^{n} \frac{(w, v_i)^2}{||v_i||^2}.
   \]
5. (8 points) Let $G$ be a group (not necessarily abelian). Suppose $\rho : G \to GL_n(\mathbb{C})$ is a homomorphism, i.e., $\rho(g_1)\rho(g_2) = \rho(g_1g_2)$ for all $g_1, g_2 \in G$. Finally, suppose the only $G$–invariant subspaces are $\{0\}$ and $\mathbb{C}^n$, i.e., if $W$ is a subspace of $\mathbb{C}^n$ and $\rho(g)W \subseteq W$ for all $g \in G$, then $W = \{0\}$ or $\mathbb{C}^n$. Show that if $A \in M_n(\mathbb{C})$ satisfies $A\rho(g) = \rho(g)A$ for all $g \in G$, then $A = cI_n$, for some $c \in \mathbb{C}$.

Hint: Find some $G$–invariant subspaces associated with $A$.

6. (12 points) Find the characteristic polynomial, minimal polynomial, and rational canonical form of the matrix

$$
\begin{pmatrix}
0 & 2 & 2 & 2 \\
2 & 0 & 2 & 2 \\
2 & 2 & 0 & 2 \\
2 & 2 & 2 & 0
\end{pmatrix}
\in M_4(\mathbb{Q}).
$$

7. (3 points) Suppose $T$ is a linear operator on $\mathbb{R}^n$, $f \in \mathbb{R}[x]$ and $\alpha$ is a (real) eigenvalue of $f(T)$. Is there a (real) eigenvalue $\beta$ of $T$ so that $f(\beta) = \alpha$? Give a proof or counterexample.

8. (5 points each) Let $F$ be a field with $p$ elements.

(a) Determine the order of the group $GL_3(F)$ of $3 \times 3$ invertible matrices with entries in $F$.

(b) Determine the order of the group $SL_3(F)$, the elements of $GL_3(F)$ of determinant $1$.

9. (4 points each) Let $V$ be a finite dimensional complex inner product space, and suppose $T$ is a normal operator on $V$.

(a) Prove $T$ is self adjoint if and only if all eigenvalues of $T$ are real.

(b) Prove $T$ is unitary if and only if all eigenvalues of $T$ have norm $1$.

10. (5 points each) Let $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. Note that $S^1$ is an abelian group with the operation of complex multiplication.

(a) Is $S^1$ finitely generated? Why or why not?

(b) Let $\chi : S^1 \to GL_1(\mathbb{C}) \simeq \mathbb{C} \setminus \{0\}$ be a $\mathbb{Z}$-linear map (i.e., a group homomorphism). Suppose $z_1, z_2, \ldots, z_k$ are elements of $S^1$ of finite orders $m_1, m_2, \ldots, m_k$, and further suppose $\gcd(m_i, m_j) = 1$, for $i \neq j$. Show there is a positive integer $n$ so that $\chi(z_i) = z_i^n$ for $i = 1, 2, \ldots, k$.

11. (5 points) Let $F$ be a field and $t_0, t_1, \ldots, t_n$ be distinct elements of $F$. Given elements $a_0, a_1, \ldots, a_n \in F$, show there is a polynomial $f \in F[x]$, with $\deg f \leq n$, so that $f(t_i) = a_i$, for $i = 0, 1, \ldots, n$.

12. (6 points) Let $F$ be a field, $A \in M_n(F)$, and let $T_A : M_n(F) \to M_n(F)$ be given by $T_A(B) = AB$. Show the minimal polynomial of $T_A$ is the minimal polynomial of $A$. Are the characteristic polynomials of $A$ and $T_A$ equal as well?