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Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

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1. (20 pts) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space $V$ and let $R = T(V)$ denote the range of $T$.

(a) Prove that $R$ has a complementary $T$-invariant subspace if and only if $R$ is independent of the null space $N$ of $T$, i.e., $R \cap N = 0$.

(b) If $R$ and $N$ are independent, prove that $N$ is the unique $T$-invariant subspace of $V$ that is complementary to $R$. 

2. (20 pts) Let $V$ be a 5-dimensional vector space over a field $F$ and let $T : V \to V$ be a linear operator.

(a) Prove that $V$ is the direct sum of its two subspaces $\text{Ker} T^5 = \text{the null space of } T^5$ and $\text{Im} T^5 = T^5(V)$, the range of $T^5$.

(b) Give an example of a linear operator $T$ such that $V$ is not the direct sum of its subspaces $\text{Ker} T$ and $\text{Im} T$. 
3. (14 pts) Let $n$ be a positive integer, let $V$ be an $n$-dimensional vector space over a field and let $T : V \to V$ be a linear operator. Prove or disprove that

$$\text{rank } T + \text{rank } T^3 \geq 2 \text{rank } T^2.$$
4. (16 pts) Let $F$ be a field of characteristic zero and let $V$ be a finite-dimensional vector space over $F$. Recall that a linear operator $E : V \to V$ is a projection operator on $V$ if $E^2 = E$. Assume that $E_1, \ldots, E_k$ are projection operators on $V$ and that $E_1 + \cdots + E_k = I$, the identity operator on $V$. Prove that $E_i E_j = 0$ for $i \neq j$. 
5. (14 pts) Classify up to similarity all $3 \times 3$ complex matrices $A$ such that $A^3 = I$, the identity matrix. How many equivalence classes are there?
6. (16 pts) Let $V$ be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator. Prove that $T$ is self-adjoint if and only if $\langle T\alpha, \alpha \rangle$ is real for every $\alpha \in V$. 
7. (20 pts) Let \( p \) be a prime integer and let \( F = \mathbb{Z}/p\mathbb{Z} \) be the field with \( p \) elements. Let \( V \) be a vector space over \( F \) and \( T : V \to V \) a linear operator. Assume that \( T \) has characteristic polynomial \( x^3 \) and minimal polynomial \( x^2 \).

(a) Express \( V \) as a direct sum of cyclic \( F[x]-\)modules.

(b) How many 1-dimensional \( T \)-invariant subspaces does \( V \) have?

(c) How many of the 1-dimensional \( T \)-invariant subspaces of \( V \) are direct summands of \( V \)?

(d) How many 2-dimensional \( T \)-invariant subspaces does \( V \) have?

(e) How many of the 2-dimensional \( T \)-invariant subspaces of \( V \) are direct summands of \( V \)?
8. (14 pts) Let $M$ be a module over the integral domain $D$. Recall that a submodule $N$ of $M$ is said to be pure if the following holds: whenever $y \in N$ and $a \in D$ are such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) If $N$ is a direct summand of $M$, prove that $N$ is pure in $M$.

(b) For $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of $x$ in the quotient module $M/N$. If $N$ is a pure submodule of $M$, and $\text{ann} \overline{x} = \{a \in D \mid ax = 0\}$ is the principal ideal $(d)$ of $D$, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\text{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal $(d)$.
9. (12 pts) Let $M$ be a finitely generated module over the polynomial ring $F[x]$, where $F$ is a field, and let $N$ be a pure submodule of $M$. Prove that there exists a submodule $L$ of $M$ such that $N + L = M$ and $N \cap L = 0$. 
10. (14 pts) Let $D$ be a principal ideal domain, let $n$ be a positive integer, and let $D^{(n)}$ denote a free $D$-module of rank $n$.

(a) If $L$ is a submodule of $D^{(n)}$, prove that $L$ is a free $D$-module of rank $m \leq n$.

(b) If $L$ is a proper submodule of $D^{(n)}$, prove or disprove that the rank of $L$ must be less than $n$. 

11. (7 pts) Let $V$ be a 5-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator such that $\text{rank } T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of $T$. Explain.

12. (5 pts) List up to isomorphism all abelian groups of order 24.
13. (16 pts) Let $V$ be an abelian group generated by elements $a, b, c$. Assume that $2a = 4b$, $2b = 4c$, $2c = 4a$, and that these three relations generate all the relations on $a, b, c$.

(a) Write down a relation matrix for $V$.

(b) Find generators $x, y, z$ for $V$ such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of the cyclic subgroups generated by $x, y, z$.

(c) Express your generators $x, y, z$ in terms of $a, b, c$.

(d) What is the order of $V$?
14. (4 pts) State true or false and justify: If $N_1$ and $N_2$ are $4 \times 4$ nilpotent matrices over the field $F$ and if $N_1$ and $N_2$ have the same minimal polynomial, then $N_1$ and $N_2$ are similar.

15. (4 pts) State true or false and justify: If $A$ and $B$ are $n \times n$ matrices over a field $F$, then $AB$ and $BA$ have the same minimal polynomial.

16. (4 pts) State true or false and justify: If $V$ is a finite-dimensional vector space and $W_1$ and $W_2$ are subspaces of $V$ such that $V = W_1 \oplus W_2$, then for any subspace $W$ of $V$ we have $W = (W \cap W_1) \oplus (W \cap W_2)$. 