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Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

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Notation: Let $F$ be a field, let $n$ be a positive integer, and let $V$ be an $n$-dimensional vector space over $F$. Let $S$ and $T$ be linear operators on $V$.

1. (13 pts) If $T$ has $n$ distinct characteristic values and $S$ commutes with $T$, prove that there exists a polynomial $f(t) \in F[t]$ such that $S = f(T)$.

2. (7 pts) Prove or disprove: If $S$ commutes with $T$ and $a \in F$, then the null space of $T - aI$ is invariant for $S$. 
Notation: If $K$ is a commutative ring and $m$ and $n$ are positive integers, then $K^{m \times n}$ denotes the $K$-module of $m \times n$ matrices with entries in $K$.

3. (6 pts) State true or false and justify: If $\mathcal{F} \subset \mathbb{C}^{4 \times 4}$ is a subspace of commuting matrices, then $\dim \mathcal{F} \leq 4$.

4. (12 pts) Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relation matrix for $V$ as a $\mathbb{Z}$-module.

(b) Let $W$ be the cyclic subgroup of $V$ generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$. Write down a relation matrix for $W$.

(c) Write down a relation matrix for the quotient module $V/W$. 
5. Let $K$ be a commutative ring with identity, $n$ a positive integer, and let $D : K^{n \times n} \to K$ be a function.

(a) (3 pts) Define “$D$ is $n$-linear”.

(b) (3 pts) If $D$ is $n$-linear, define “$D$ is alternating”.

(c) (3 pts) Define “$D$ is a determinant function.”

(d) (4 pts) If $n = 3$ and $K$ is a field, what is the dimension of the $K$-vector space of all 3-linear functions on $K^{3 \times 3}$?

(e) (5 pts) If $K$ is the polynomial ring $\mathbb{Q}[\{x_{ij}\}]$, where $1 \leq i \leq 5$, $1 \leq j \leq 5$, and $A = (x_{ij}) \in K^{5 \times 5}$, then $\det A$ is a sum of monomials in the $x_{ij}$. How many terms are in this sum? Explain.
6. (8 pts) Let $V$ be an $n$-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator. Assume that $c \in F$ is such that there exists a nonzero vector $\alpha$ with $T\alpha = c\alpha$. Prove that there exists a nonzero linear functional $f$ on $V$ such that $T^t f = cf$, where $T^t$ is the transpose of $T$.

7. (8 pts) Let $F$ be a field and let $L$ be a linear functional on the polynomial ring $F[x]$ having the property that $L(fg) = L(f)L(g)$ for all polynomials $f, g \in F[x]$. Prove that either $L = 0$ or there exists $c \in F$ such that $L(f) = f(c)$ for all $f \in F[x]$. 
8. Let $V$ be a finite-dimensional vector space over a field $F$, let $T : V \to V$ be a linear operator, and let $p(x) \in F[x]$ be the minimal polynomial of $T$. Assume that $p(x) = p_1^{r_1} \cdots p_k^{r_k}$, where the $p_i \in F[x]$ are distinct monic irreducible polynomials, $i = 1, \ldots, k$, and the $r_i$ are positive integers. Let $W_i = \{ v \in V \mid p_i(T)^{r_i}(v) = 0 \}$.

(a) (8 pts) Describe how to obtain linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each $i$, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on $V$.

(b) (8 pts) If $p(x)$ is a product of linear polynomials, describe how to obtain a diagonalizable operator $D$ and a nilpotent operator $N$ such that $T = D + N$, where $D$ and $N$ are both polynomials in $T$. 
9. (8 pts) Prove or disprove: if \( V \) is a vector space over a field \( F \) and \( T : V \to V \) is a linear operator such that every subspace of \( V \) is invariant under \( T \), then \( T \) is a scalar multiple of the identity operator.

10. Let \( F \) be a field and let \( g(x) \in F[x] \) be a monic polynomial.

   (a) (4 pts) Describe the \( F[x] \)-submodules of \( V = F[x]/(g(x)) \).

   (b) (5 pts) If \( g(x) = x^3(x - 1) \), diagram the lattice of \( F[x] \)-submodules of \( V = F[x]/(g(x)) \).
11. (16 pts) Let $D$ be a principal ideal domain and let $V$ and $W$ denote free $D$-modules of rank 3 and 4, respectively. Assume that $\phi : V \to W$ is a $D$-module homomorphism, and that $B = \{v_1, v_2, v_3\}$ is an ordered basis of $V$ and $B' = \{w_1, w_2, w_3, w_4\}$ is an ordered basis of $W$.

(a) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis $B$.

(b) Describe how to obtain a matrix $A \in D^{4 \times 3}$ so that left multiplication by $A$ on $D^3$ represents $\phi : V \to W$ with respect to $B$ and $B'$.

(c) How does the matrix $A$ change if we change the basis $B'$ by replacing $w_2$ by $w_2 + aw_1$ for some $a \in D$?

(d) How does the matrix $A$ change if we change the basis $B$ by replacing $v_2$ by $v_2 + av_1$ for some $a \in D$?
12. (20 pts) Let $p$ be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with $p$ elements. Let $V$ be a vector space over $F$ and $T : V \to V$ a linear operator. Assume that $T$ has characteristic polynomial $x^4$ and minimal polynomial $x^3$.

(a) Express $V$ as a direct sum of cyclic $F[x]$-modules.

(b) How many 3-dimensional cyclic $T$-invariant subspaces does $V$ have?

(c) How many of the 3-dimensional cyclic $T$-invariant subspaces of $V$ are direct summands of $V$?

(d) How many noncyclic 3-dimensional $T$-invariant subspaces does $V$ have?

(e) How many of the noncyclic 3-dimensional $T$-invariant subspaces of $V$ are direct summands of $V$?
13. (14 pts) Let $V$ be an abelian group generated by elements $a, b, c$. Assume that $3a = 6b$, $3b = 6c$, $3c = 6a$, and that these three relations generate all the relations on $a, b, c$.

(a) What is the order of $V$? Justify your answer.

(b) What is the order of the element $a$? Justify your answer.
14. (10 pts) Let $V$ be a vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many of its proper subspaces.

15. (6 pts) Let $F$ be a finite field with $|F| = q$, and let $G = \{ A \in F^{3\times3} \mid \det A \neq 0 \}$.

(a) What is $|G|$?

(b) Let $H = \{ A \in G \mid \det A = 1 \}$. What is $|H|$?
16. (6 pts) Let $A \in \mathbb{R}^{n \times n}$ and let $f_1, \ldots, f_n$ be the diagonal entries in the normal form of $xI - A$.

(i) For which matrices $A$ is $f_1 \neq 1$?

(ii) For which matrices $A$ is $f_{n-1} = 1$?

17. (9 pts) Let $A \in \mathbb{R}^{3 \times 3}$ be such that $\det A = 3$ and let $\text{adj}(A) \in \mathbb{R}^{3 \times 3}$ denote the classical adjoint of $A$.

(a) What is the product $\text{adj}(A)A$?

(b) What is $\det(\text{adj} A)$?

(c) What is $\text{adj}(\text{adj}A)$?
18. (6 pts) Let $V$ be a 4-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator such that $\text{rank } T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of $T$. Explain.

19. (8 pts) Let $F$ be a field and let $V = F^{4 \times 4}$. Let $W$ be the subspace of $V$ spanned by all matrices of the form $C = AB - BA$, where $A, B \in V$. Prove that $W$ is the subspace of $V$ of matrices having trace zero.