1. Let $A \in M_{n \times n}(K)$ (the vector space of $n \times n$ matrices over a field $K$). State and prove the Cayley-Hamilton Theorem for $A$. :
2. Let $S = R[x]^3/(f_1, f_2, f_3)$ be a module over $R[x]$ where $R[x]$ is the ring of real polynomials and $f_1 = (x, 1, 0)$, $f_2 = (1, x, 0)$, $f_3 = (0, 0, x - 1)$ be three elements in $R[x]^3$. Express $R[x]^3/(f_1, f_2, f_3)$ as direct sum of modules $\bigoplus_{i=1}^{m} R[x]/(c_i)$ such that $m \leq 3$ and $c_i | c_{i+1}$. (The fundamental theorem of finitely generated modules over P.I.D.)
3. Show that the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective for integer $n \geq 2$. 
4. Show that $\text{Ext}^1_Z(Z/mZ, Z) \approx Z/mZ$ for $m \geq 2$. 
5. Find the best straight line fit (least square approximation) to the measurement $b = 1$ at $t = 0$, $b = 3$ at $t = 1$, $b = 3$ at $t = 2$. 
6. Find an orthonormal basis for $P_3$, the vector space of all polynomials of degree $\leq 3$ under the inner product defined as

$$< f | g > = \int_0^1 f g \, dx$$
7. Let $R$ be the field of real numbers. Let $W$ be the subspace of $R^4$ generated by $(1, 1, 0, 0)^T$, $(0, 0, 1, 1)^T$. Given $x = (1, 2, 3, 4)^T$. Find $y, z \in R^4$ such that $x = y + z$ and $y \in W, z \in W^\perp$. 
8. Find an $2 \times 2$ matrix $A$ which has all the principal minors positive and which is not a positive matrix.
9. Let $A$ be the matrix over complex numbers as follows,

$$A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 2 \\
2 & 2 & 2
\end{pmatrix}.$$ 

Find the Jordan canonical form of $A$. 
10. Decide if the following function has a local minima at the origin;

\[ f(x, y, z) = 3x^2 + 6xy + 2xz + 4y^2 + 3yz + 2z^2 \]