QUALIFYING EXAMINATION

August 2014

MA 554

1. (12 points) Let V and W be vector spaces over a field K and let $\varphi : V \to W$ be a K-linear map. Let $\varphi^* : W^* \to V^*$ denote the induced map between the dual spaces, which is defined by $\varphi^*(f) = f \circ \varphi$.

(a) Prove that if φ is surjective, then φ^* is injective.

(b) Prove that if φ is injectve, then φ^* is surjective.

2. (14 points) Let A be an n by n matrix with integer coefficients and let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ be the $\mathbb{Z}$-linear map defined by $\varphi(x) = A \cdot x$, where $x \in \mathbb{Z}^n$ is considered as a column vector. Prove that φ is surjective if and only if $\det(A) = 1$ or $\det(A) = -1$.

3. (14 points) Consider the matrix

$$A = \begin{pmatrix} 8 & -2 & -5 & 3 \\ -6 & 2 & 2 & -4 \\ 10 & -4 & -7 & 3 \end{pmatrix}$$

with entries in the ring $\mathbb{Z}$. Let G be the cokernel of A, i.e., the Abelian group with 3 generators and 4 generating relations given by the 4 columns of A.

(a) Determine the order of G.

(b) Determine the smallest positive integers n with $n \cdot G = 0$.

(c) Determine the smallest number m so that G can be generated by m elements.

4. (15 points) Let $\mathrm{M}_n(\mathbb{C})$ be the space of n by n matrices with entries in $\mathbb{C}$, let $A \in \mathrm{M}_n(\mathbb{C})$, and let V be the $\mathbb{C}$-subspace of $\mathrm{M}_n(\mathbb{C})$ spanned by $\{A^k | k \geq 0\}$.

(a) Prove that $\dim_{\mathbb{C}} V \leq n$.

(b) Give a criterion, in terms of the Jordan canonical form of A, for when equality holds in part (a).

5. (14 points) Let A and B be 6 by 6 matrices with entries in $\mathbb{R}$ that do not have any real eigenvalues. Show that A and B are similar if and only if they have the same characteristic polynomials and the same minimal polynomials.

6. (15 points) Let V be a finite-dimensional vector space over a field K, let $\varphi : V \to V$ be a K-linear map, and let U be a φ-invariant subspace of V. Show that if there exists a vector $v \in V$ and a positive integer n so that $\{v, \varphi(v), \ldots, \varphi^{n-1}(v)\}$ is a basis of V, then there exists a vector $u \in U$ and a positive integer m so that $\{u, \varphi(u), \ldots, \varphi^{m-1}(u)\}$ is a basis of U.

7. (16 points) Let V be a finite-dimensional inner product space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$, and let $\varphi : V \to V$ be a K-linear map satisfying $\varphi^5 = \varphi^3$. Prove that φ is normal if and only if it is self-adjoint.