1. Let \( S^2 = \{(x, y, z); x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 \) be the unit sphere. Denote the north pole by \( N = (0, 0, 1) \) and the south pole by \( S = (0, 0, -1) \). Denote \( U_1 = S^2 \setminus \{N\} \) and \( U_2 = S^2 \setminus \{S\} \). For any \( p \in S^2 \setminus \{N\} \), denote by \( p\overrightarrow{N} \) the line in \( \mathbb{R}^3 \) passing through \( p \) and \( N \). Use similar notation for \( p\overrightarrow{S} \).

The stereographic projection \( \Phi_1 : U_1 = S^2 \setminus \{N\} \to \mathbb{R}^2 = \{(x, y, 0); (x, y) \in \mathbb{R}^2\} \) is defined as:

\[
\Phi_1(p) = \overrightarrow{pN} \cap \{z = 0\}, \quad \text{for any} \ p \in U_1.
\]

Similarly \( \Phi_2 : U_2 := S^2 \setminus \{S\} \to \mathbb{R}^2 \) is defined as

\[
\Phi_2(p) = \overrightarrow{pS} \cap \{z = 0\}, \quad \text{for any} \ p \in U_2.
\]

Prove that \( (U_1 = S^2 \setminus \{N\}, \Phi_1), (U_2 = S^2 \setminus \{S\}, \Phi_2) \) define a differentiable structure on \( S^2 \). What is the transition function between these two coordinate charts?

2. Is there a smooth vector field on the 2-dimensional torus \( S^1 \times S^1 \) with a single zero point? What is the index of that zero point if such a vector field exists?

3. Denote by \( M = \text{Mat}_{3 \times 2}(\mathbb{R}) \cong \mathbb{R}^6 \) the set of \( 3 \times 2 \) real matrices. \( A^T \) denotes the transpose of \( A \). Prove that the set \( N = \{A \in \text{Mat}_{3 \times 2}(\mathbb{R}); A^TA = I_2\} \) is a smooth manifold. What is its dimension?

4. Prove the following result:

Let \( M \) be a smooth manifold. A smooth 1-form \( \omega \) on \( M \) is an exact form if and only if, for any closed piece-wisely smooth curve \( C \) on \( M \), \( \int_C \omega = 0 \).

5. Let \( i : M \to N \) be an immersed submanifold and \( X \) be a smooth vector field on \( M \).

(a) If \( M \) is a smooth embedded submanifold, prove that there exists a smooth vector field \( X \) on \( N \) such that \( X_p = i_*(X_p) \) for any \( p \in M \).

(b) Is the above statement true if \( i \) is only an injective immersion but not an embedding?

6. Consider the distribution \( \Delta \) on \( \mathbb{R}^3 \) defined as the kernel of \( \alpha \) where:

\[
\alpha = ydx - xdy + dz.
\]

In other words, \( \Delta_p = \text{Ker}(\alpha_p) \) for any \( p \in \mathbb{R}^3 \). Is the distribution \( \Delta \) integrable?