Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded.

**Problem 1.** [15 points]
Let $M(n, \mathbb{R})$ be the set of all $n \times n$ matrices (this is a manifold diffeomorphic to $\mathbb{R}^{n^2}$). Let $M_k(n, \mathbb{R})$ denote the subset of all rank $k$ matrices. Prove that $M_k(n, \mathbb{R})$ is a submanifold and find its dimension.

**Problem 2.** [15 points]
Let $f: \mathbb{R}^5 \to \mathbb{R}^3$ be a smooth map. Prove that there exists a sphere $S \subset \mathbb{R}^3$ centered about the origin such that $f^{-1}(S)$ is a smooth submanifold of $\mathbb{R}^5$.

**Problem 3.** [15 points]
Let $X, Y$ be compact, oriented $n$–manifolds without boundary and assume that $Y$ is connected. Prove that if $f: X \to Y$ is a smooth function, then

$$\deg(f) = I(\text{Graph}(f), X \times \{y\})$$

for any $y \in Y$.

**Problem 4.** [15 points]
Let $S^2 \subset \mathbb{R}^3$ be the standard 2–sphere and $i: S^2 \to \mathbb{R}^3$ the inclusion map. Define

$$\omega = (x^2 + x + y)dy \wedge dz.$$

(a) Calculate

$$\int_{S^2} \omega.$$

State which orientation you are using.

(b) Prove or disprove: there exists a closed form $\alpha \in \Omega^2(\mathbb{R}^3)$ such that $i^*(\alpha) = i^*(\omega)$.

**Problem 5.** [15 points]
Let $M, N$ be compact, oriented manifolds of dimension $m, n$, respectively. Orient $M \times N$ with the product orientation and let $P_M, P_N: M \times N \to M, N$ be the projection maps onto $M, N$, respectively. Prove that if $\omega \in \Omega^m(M)$ and $\eta \in \Omega^n(N)$ that

$$\int_{M \times N} P_M^*(\omega) \wedge P_N^*(\eta) = \int_M \omega \cdot \int_N \eta.$$