Math 562: Summer 2016 Qualifying Exam (McReynolds)

PUID Number:____

Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded.

Problem 1. [15 points]

Let M be a smooth connected manifold without boundary. We say $p \sim q$ for $p,q \in M$ if there exists a diffeomorphism $f \colon M \longrightarrow M$ such that $f(p) = q$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Prove that for any $p,q \in M$, there exists a diffeomorphism $f \colon M \to M$ such that $f(p) = q$.

Problem 2. [15 points]

Prove that if $f \colon X \to Y$ is a smooth function with f transverse to a submanifold Z of Y and $S \subset X$ is a submanifold such that S and $f^{-1}(Z)$ are transverse, then $f_{|S}$ is transverse to Z.

Problem 3. [15 points]

Let Y be a compact, orientable manifold with compact n–dimensional submanifolds X, Z such that $X \cup Z = Y$ and $X \cap Z$ is a $(n-1)$–dimensional submanifold. Prove that $\chi(Y) = \chi(X) + \chi(Z) - \chi(X \cap Z)$.

Problem 4. [15 points]

Let $f \colon \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x,y) = (x^2 + y, y^2 - x)$. Compute $df_p, f^*(dx), f^*(dy)$, and $f^*(dx \wedge dy)$.

Problem 5. [15 points]

(a) Let M be a compact, orientable $(k+1)$–manifold with boundary ∂M. Let $f \colon \partial M \to N$ be a smooth map and $\omega \in \Omega^k(N)$ be a closed form. Prove that if f can be extended to a smooth map $F \colon M \to N$, then
$$\int_{\partial M} f^*(\omega) = 0.$$

(b) Let S^1 be the unit circle in $\mathbf{R}^2$ and D^2 the unit disk with $\partial D^2 = S^1$. Prove that the smooth function $f \colon S^1 \to S^1$ given by $f(x,y) = (-x,-y)$ can not be smoothly extended to D^2.