1. Prove that a $d$-dimensional manifold $X$, for which there exists an immersion $f : X \to \mathbb{R}^{d+1}$, is orientable if and only if there is a smooth nowhere vanishing normal vector field along $(X, f)$.

2. Define $\omega = \frac{-y \, dx}{x^2 + y^2} + \frac{x \, dy}{x^2 + y^2}$. Calculate $\int_\gamma \omega$, where $\gamma$ is the curve $x^8 + y^8 = 1$, oriented counterclockwise.

3. Let $f(x, y, z) = x^2 y + e^x + z$. Show that there exists a differentiable function $g(y, z)$, defined near $(y, z) = (1, -1)$, so that $g(1, -1) = 0$ and $f(g(y, z), y, z) = 0$.

4. Prove that the set of all $3 \times 3$ matrices of the form

$$
\begin{pmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix}
$$

is a Lie group and that the exponential mapping in $G$ maps $T_e G$ in a one-one manner globally onto $G$.

5. Let $M$ be a compact manifold and $f : M \to \mathbb{R}$, a $C^1$ function. Show that there exist at least two points where $df = 0$. Give an example with exactly two points.

6. Suppose $p \leq d$ and let $\omega_1, \omega_2, \ldots, \omega_p$ be linearly independent 1-forms on $M^d$ such that for some $\theta_1, \theta_2, \ldots, \theta_p$, $\sum_{i=1}^{p} \theta_i \wedge \omega_i = 0$. Show $\theta = \sum_{j=1}^{p} A_{ij} \omega_j$ for $C^\infty$ functions $A_{ij}$, satisfying $A_{ij} = A_{ji}$.

7. Show that $S^k \times S^\ell$ can be embedded in $\mathbb{R}^{k+\ell+1}$.

8. Is the open ball $B^n$ in $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^n$?

9. Define $\zeta = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{r^3}$ in $\mathbb{R}^3 - 0$.
   a. Show $d\zeta = 0$
   b. Is $\zeta$ exact in $\mathbb{R}^3 - 0$?
   c. Is $\zeta$ exact in the complement of each line through 0?

10. Prove that the unit tangent bundle of $S^2$ is diffeomorphic to $SO(3)$. 