1. The hyperbolic space $H^n$ is the unit disc $\{(x_1, \ldots, x_n)| \sum_1^n x_i^2 < 1\}$ with the hyperbolic metric $g_H = \left(\frac{2}{1-\sum_1^n x_i^2}\right)^2 g_E$, where $g_E$ is the Euclidean metric. Consider the parametrized surface $S = F(D^2) \subset H^n$, where $F(u,v) := (u,v - u,v)$ and $D^2 = \{(u,v)| u^2 + v^2 < 1/4\}$. Let $\Omega$ be the volume form on $S$ associated to the metric induced from the hyperbolic metric, and the orientation determined by the parametrization.

(a) Express $F^{\ast}\Omega$ in $du, dv$.
(b) Use the Stokes’ theorem to express the area of $S$ as a line integral over $\partial D$.

2. Let $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ be the Lie group of $2 \times 2$ real matrices of determinant 1. Let $x_{ij}$ denote the global coordinate function on $GL(2, \mathbb{R})$ which sends a matrix to its $ij$-th entry. Let $e$ denote the identity.

(a) Let $v$ be a left invariant vector field on $GL(2, \mathbb{R})$. Then $v(g) = \sum_{ij} f_{ij}(g) \frac{\partial}{\partial x_{ij}}$, where $f_{ij}$ is a function on $GL(2, \mathbb{R})$. Express the matrix $(f_{ij}(g))$ in terms of the matrix $(f_{ij}(e))$ for a general $g \in GL(2, \mathbb{R})$.
(b) It is known that by sending $v$ to the matrix $(f_{ij}(e))$ above, the Lie algebra of $GL(2, \mathbb{R})$ can be identified with the Lie algebra of $2 \times 2$ matrices and commutators, and hence the Lie algebra of $SL(2, \mathbb{R})$ is identified with a Lie subalgebra of the latter. Show that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an element in this Lie subalgebra, and find the 1-dimensional subgroup of $SL(2, \mathbb{R})$ which is an integral manifold of the corresponding invariant vector field.

3. Let $T^2 = (\mathbb{R}/\mathbb{Z})^2$. Show that $\Omega^\ast(T^2)$ is a free module over $C^{\infty}(T^2)$, and compute the de Rham cohomology groups of $T^2$.

4. Let $M$ be a manifold of dimension $m$ and $\alpha$ be a nowhere vanishing 1-form on $M$. Let $\xi_\alpha$ denote the $(m-1)$-plane distribution $\ker \alpha$, regarding $\alpha(x)$ as a linear map from $T_x M$ to $\mathbb{R}$.

(a) Show that $\xi_\alpha$ is integrable iff $\alpha \wedge d\alpha = 0$.
(b) Suppose $m = 2n + 1$ is odd. Show that the following two conditions are equivalent:
   (i) The bilinear form $d\alpha$ on $T_x M \times T_x M$ is nondegenerate when restricted on $\xi_\alpha(x) \times \xi_\alpha(x)$ for every $x \in M$;
   (ii) $\alpha \wedge (d\alpha)^n$ is nowhere vanishing.
   (Hint: Assume the following fact from linear algebra: If $V$ is a finite dimensional vector space and $w$ is a nondegenerate skew-symmetric bilinear form on $V \times V$, then there is a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$ of $V$, such that $w(u_i, u_j) = w(v_i, v_j) = 0$, $w(u_i, v_j) = \delta_{ij} \forall i, j$)
   (c) A 1-form $\alpha$ satisfying the conditions in (b) above is called a contact form. Let $N$ be a Riemannian manifold. Show that the 1-form $\alpha_{can}$ on $T^* N$ defined below restricts to a contact form on the unit cotangent bundle of $N$: Let $V$ be a tangent vector at $(x, p) \in T^* N$, where $x \in N$ and $p \in T_x^* N$, and let $\pi : T^* N \rightarrow N$ be the projection map. Then the pairing $\langle \alpha_{can}(x, p), V \rangle := \langle p, \pi_V V \rangle$. 

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5. Let $M$ be a compact connected Riemannian manifold, and $f : M \to S^1 = \mathbb{R}/\mathbb{Z}$ be a submersion. For $t \in S^1$, denote the level hypersurfaces $f^{-1}(t)$ by $N_t$.

(a) Show that $N_t$ are embedded submanifolds, and they are orientable iff $M$ is.

(b) Let $v_f$ denote the vector field dual to the 1-form $df$. Show that $v_f$ is nowhere vanishing and complete, and that $f$ is surjective.

(c) Show that $M$ can not be simply connected.