(1) (a) Prove that
\[ M := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\} \]
is a smooth manifold. Determine its dimension.

(b) Does \( M \) have boundary? If yes, determine \( \partial M \).

(2) Let \( M \) be a manifold without boundary.

(a) Give one definition of \( T_x M, x \in M \).

(b) Let \( f : M \to \mathbb{R} \) be a smooth function. Define \( Df(x) : T_x M \to \mathbb{R} \).

(c) Let \( m \in M \) be a maximum of \( f \), that is, \( f(x) \leq f(m) \) for all \( x \in M \).

Prove that \( Df(m) : T_m M \to \mathbb{R} \) vanishes.

(3) We denote by \( M(n) \) the vector space of all \( n \times n \) matrices. Let \( O(n) \) be the orthogonal group, that is
\[ O(n) := \{ A \in M(n) \mid AA^t = I \} . \]

(a) Prove that \( O(n) \) is a manifold. Determine its dimension.

(b) Give an explicit description of \( T_1 O(n) \).

(4) We denote by \( \mathbb{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 100\} \) the closed disk of radius 10. Let \( v : \mathbb{D}^2 \to \mathbb{R}^2 \) be the vector field given by
\[ v(x, y) = (p(x)e^x, q(y)e^y) \]
where \( p(x) = x^3 - x \) and \( q(y) = y^3 + 3y^2 + 2y \).
Compute the index \( \text{ind}(v) \).
(5) Let $X = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} - z \frac{\partial}{\partial x}$ be vector fields on $\mathbb{R}^3$ with respect to the standard coordinates $(x,y,z) \in \mathbb{R}^3$.
Compute $[X,Y]$.

(6) Let $G$ be a Lie group. We denote by $L_g$ resp. $R_g$ left resp. right multiplication with $g \in G$.

(a) Give the definition of a left invariant vector field $X$ and prove that $(R_g)_*X$ is also left invariant.

(b) Use part (a) to prove that the map $Ad(g) : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$Ad(g)X := (R_g^{-1})_*X$$

is well-defined. That is, prove that $X \in \mathfrak{g}$ implies $Ad(g)X \in \mathfrak{g}$.

(c) State the definition of the exponential map $\exp : \mathfrak{g} \rightarrow G$ in terms of the flow of a left invariant vector field. Prove

$$(Ad(g)X)(e) = \frac{d}{dt}\bigg|_{t=0} (g \cdot \exp(tX) \cdot g^{-1}) .$$

(7) We consider $\mathbb{R}^6$ with coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)$ and define the 1-form

$$\lambda = \sum_{i=1}^{3} x_i \, dy_i$$

(a) Compute $\omega := d\lambda$.

(b) Compute $\omega \wedge \omega \wedge \omega$. Simplify all expressions!

(c) Show that $\omega \wedge \omega$ is exact, that is, find a 3-form $\Lambda$ with $\omega \wedge \omega = d\Lambda$. 