1. Let \( C \) be the unit circle in \( \mathbb{R}^2 \) and \( S \) the boundary of the unit square centered at the origin. Show that there is no diffeomorphism \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) with \( F(C) = S \).

2. Suppose that \( V \) is a 3-dimensional vector space with basis \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \). Show that \( V \) may be given the structure of a Lie algebra so that \( [\vec{v}_1, \vec{v}_2] = \vec{v}_3, [\vec{v}_1, \vec{v}_3] = [\vec{v}_2, \vec{v}_3] = 0 \). Prove that every two dimensional subalgebra contains \( \vec{v}_3 \).

3. Does there exist a \( C^\infty \) vector field on \( S^n \) which vanishes at a) exactly two points, b) exactly one point?

4. Let \( N \) be a submanifold contained in a manifold \( M \). Suppose \( \gamma: (a, b) \to M \) is a \( C^\infty \) curve such that \( \gamma(a, b) \subset N \). Show by example that it is not necessarily true that \( \dot{\gamma}(t) \in T_{\gamma(t)}N \), for each \( t \in (a, b) \).

5. Define \( \omega = (x + y)dz - (y + z)dx + (x + z)dy \) and suppose \( S \) denotes the set where \( x^2 + y^2 + z^2 = 1 \) and \( z \geq 0 \). Evaluate \( \int_{\partial S} \omega \) both directly and by Stokes’ theorem.

6. Suppose that \( S^* = S^3 - (0,0,0,1) \), where \( S^3 \) denotes the three sphere \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \). Define vector fields \( V \) and \( W \) by \( V = (1 - x_4 - x_1^2, -x_1x_2, -x_1x_3, x_1(1 - x_4)) \) and \( W = (-x_1x_2, 1 - x_4 - x_2^2, -x_2x_3, x_2(1 - x_4)) \). Show that \( V \) and \( W \) are tangent to \( S^* \) and linearly independent.

7. If \( k \) is a real number, show that a nonempty subset \( T^k \) of \( S^* \), defined by \( x_3 + kx_4 = k \), \( kx_3 - x_4 \neq 0 \), is a two dimensional submanifold. Here one uses the notation of Problem 6. Is the inequality \( kx_3 - x_4 \neq 0 \) a consequence of the other hypotheses?

8. Show that each \( T^k \) is an integral manifold of the distribution spanned by \( V \) and \( W \). Is this an involutive distribution on all of \( S^* \)?

9. If two maps \( f \) and \( g \) from \( X \) to \( S^p \) satisfy \( ||f(x) - g(x)|| < 2 \), for all \( x \), prove that \( f \) is homotopic to \( g \), the homotopy being smooth if \( f \) and \( g \) are smooth.

10. Suppose that \( p \) denotes the distance from the center of the ellipsoid \( \Sigma, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \), to the tangent plane at the point \( P(x, y, z) \). Compute \( \iint_{\Sigma} pdS \) and \( \iint_{\Sigma} \frac{1}{p} dS \), where \( dS \) is the area element.