Qualifying Exam
August 2011

Math 562
Professor Donnelly

1. Let C be the unit circle in R^2 and S the boundary of the unit square centered at the origin. Show that there is no diffeomorphism $F\colon R^2 \to R^2$ with $F(C) = S$.

2. Suppose that V is a 3–dimensional vector space with basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Show that V may be given the structure of a Lie algebra so that $[\vec{v}_1, \vec{v}_2] = \vec{v}_3, [\vec{v}_1, \vec{v}_3] = [\vec{v}_2, \vec{v}_3] = 0$. Prove that every two dimensional subalgebra contains $\vec{v}_3$.

3. Does there exist a C^∞ vector field on S^n which vanishes at a) exactly two points, b) exactly one point?

4. Let N be a submanifold contained in a manifold M. Suppose $\gamma\colon (a, b) \to M$ is a C^∞ curve such that $\gamma(a, b) \subset N$. Show by example that it is not necessarily true that $\dot{\gamma}(t) \in T_{\gamma(t)}N$, for each $t \in (a, b)$.

5. Define $\omega = (x + y)dz - (y + z)dx + (x + z)dy$ and suppose S denotes the set where $x^2 + y^2 + z^2 = 1$ and $z \geq 0$. Evaluate $\int_{\partial S} \omega$ both directly and by Stokes' theorem.

6. Suppose that $S^* = S^3 - (0, 0, 0, 1)$, where S^3 denotes the three sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Define vector fields V and W by $V = (1 - x_4 - x_1^2, -x_1x_2, -x_1x_3, x_1(1 - x_4))$ and $W = (-x_1x_2, 1 - x_4 - x_2^2, -x_2x_3, x_2(1 - x_4))$. Show that V and W are tangent to S^* and linearly independent.

7. If k is a real number, show that a nonempty subset T^k of S^*, defined by $x_3 + kx_4 = k$, $kx_3 - x_4 \neq 0$, is a two dimensional submanifold. Here one uses the notation of Problem 6. Is the inequality $kx_3 - x_4 \neq 0$ a consequence of the other hypotheses?

8. Show that each T^k is an integral manifold of the distribution spanned by V and W. Is this an involutive distribution on all of S^*?

9. If two maps f and g from X to S^p satisfy $\|f(x) - g(x)\| < 2$, for all x, prove that f is homotopic to g, the homotopy being smooth if f and g are smooth.

10. Suppose that p denotes the distance from the center of the ellipsoid Σ, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, to the tangent plane at the point $P(x, y, z)$. Compute $\iint_\Sigma p dS$ and $\iint_\Sigma \frac{1}{p} dS$, where dS is the area element.