Math 562: January 2013 Qualifying Exam

PUID Number: ________________________________

Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded as otherwise this decision will be made for you with no guarantee it will be the optimal outcome.

**Problem 1.** [15 points]
Let \(X_1, X_2, Y\) be smooth manifolds and \(V_1, V_2, W\) be finite dimensional vector spaces.

(a) Let \(L_j: V_j \rightarrow W\) be linear maps for \(j = 1, 2\). Prove that if \(L_1\) is onto, then
\[
\text{Image}(L_1) \oplus \{0\} \oplus \text{Image}(L_2) + \Delta_W = W \times W
\]
where \(\Delta_W\) is the diagonal in \(W \times W\).

(b) Let \(f_j: X_j \rightarrow Y\) be smooth maps for \(j = 1, 2\). Prove that if \(f_1\) is a submersion, then
\[
Z = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}
\]
is a submanifold.

**Problem 2.** [15 points]
Let \(S^1\) be the standard unit circle in \(\mathbb{R}^2\).

(a) Let \(f: S^1 \rightarrow \mathbb{R}^3\) be a smooth function such that \(df_x\) is nonzero for all \(x \in S^1\). Prove that there exists a 2–plane \(V\) in \(\mathbb{R}^3\) with \(P_V: \mathbb{R}^3 \rightarrow V\) being the orthogonal projection map such that \(V \circ f: S^1 \rightarrow V\) has \(d(P_V \circ f)_x\) nonzero for all \(x \in S^1\).

(b) Give an example of \(f\) in (a) that is also injective but \((P_V \circ f)\) is not injective for all choices \(P_V\).

(c) Let \(F: S^1 \rightarrow \mathbb{R}^4\) be smooth and injective. Prove that there exists a 3–space \(W\) in \(\mathbb{R}^4\) with \(P_W: \mathbb{R}^4 \rightarrow W\) being orthogonal projection such that \((P_W \circ F)\) is injective.

**Problem 3.** [15 points]
Let \(S^2\) be the unit 2–sphere in \(\mathbb{R}^3\) and define
\[
F: S^2 \rightarrow S^2
\]
by
\[
F(x, y, z) = (-z, -x, -y).
\]
(a) Compute the Lefschetz number of $F$.
(b) Compute the oriented degree of $F$.

**Problem 4.** [15 points]
Let $M$ be a smooth compact $n$–manifold without boundary. Prove or disprove the following:

(a) $M$ admits a vector field with only finitely many zeroes.
(b) $M$ admits a vector field with no zeroes.

**Problem 5.** [15 points]
Let $S^2$ be the standard 2–sphere in $\mathbb{R}^3$ and
\[
\omega = (3x^2 \cos y + e^{iy})dx \wedge dy + 17x^3 dx \wedge dz + (x + yz^5 + \sin z)dy \wedge dz.
\]
Compute
\[
\int_{S^2} \omega.
\]
Please indicate what orientation you are using.