MA 571 Qualifying Exam. August 2016. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't stated in Munkres and isn't obvious, be careful to give a clear explanation.

1. Let A be a subset of a topological space X and let B be a subset of A. **Prove** that $\bar{A} - \bar{B} \subset \overline{A - B}$.

2. Let X be a compact topological space and let $\sim$ be an equivalence relation on X with the property that $X/\sim$ is Hausdorff. Let $\sim'$ be the equivalence relation on $X \times [0,1]$ defined by $(x,t) \sim' (x_1,t_1) \Leftrightarrow x \sim x_1$ and $t = t_1$. **Prove** that $(X \times [0,1])/\sim'$ is homeomorphic to $(X/\sim) \times [0,1]$.

3. **Prove** that the union of the x and y axes is not homeomorphic to $\mathbb{R}$.

4. Let X be a compact Hausdorff space and let $\sim$ be an equivalence relation on X. Suppose that the set
$$S = \{(x,y) \,|\, x \sim y\}$$
is a closed subset of $X \times X$. **Prove** that the quotient map $q : X \to X/\sim$ takes closed sets to closed sets. (Hint: for a closed subset C of X, consider $q^{-1}q(C)$).

5. **Prove** that if X is connected and locally compact Hausdorff but not compact then the one-point compactification of X is connected.

6. Let X be the quotient space obtained from an octagon P by pasting its edges together according to the labelling scheme $abab^{-1}cdc^{-1}d^{-1}$ (read the formula carefully!).

   i) Calculate $H_1(X)$. (You may use anything proved in Munkres, but be sure to be clear about what you're using.)

   ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?

7. Let $p : Y \to X$ be a covering map with X simply connected (in particular, X is path connected). Let $y_0 \in Y$ and $x_0 = p(y_0)$. Define a function $s : X \to Y$ as follows: for $x \in X$ choose a path γ from x_0 to x, lift it to $\tilde{\gamma}$ starting at y_0, and let $s(x) = \tilde{\gamma}(1)$.

   Prove that s is well-defined.

   You may use the fact (without having to prove it) that the lift of a path homotopy is again a path homotopy.