Each problem is worth 12 points. There are 96 points altogether.

1. Let $X$ be a metric space with metric $d$. Prove that $d$ is continuous as a function from $X \times X$ to $\mathbb{R}$, where $X \times X$ has the “product metric” $d'$ defined by

$$d'((x, y), (x_1, y_1)) = \max(d(x, x_1), d(y, y_1))$$

(you may assume $d'$ is indeed a metric, you do not have to prove it).

2. Let $X$ be an ordered set with the order topology. If $X$ is connected, prove that $X$ has the least upper bound property.

3. Let $X$ be the closed unit disk in $\mathbb{R}^2$, and define an equivalence relation on $X$ by: $a \sim b$ if either $a = b$ or $a$ and $b$ are both on the unit circle. Let $X^*$ be the partition of $X$ into the equivalence classes of $\sim$. Show that $X^*$, with its quotient topology, is homeomorphic to the one-point compactification of the open unit disk.

4. Show that every locally compact Hausdorff space is regular.

5. A manifold is a Hausdorff space with the property that every point has a neighborhood homeomorphic to $\mathbb{R}^n$. Prove that every manifold is regular.

6. Prove that every compact Hausdorff space is a Baire space.

7. Let $p : E \to B$ be a covering map with $B$ connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. Prove that, for every $b \in B$, $p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

8. Let $X$ be a topological space and $f : [0, 1] \to X$ any continuous function. Define $\tilde{f}$ by $\tilde{f}(t) = f(1 - t)$. Prove that $f \ast \tilde{f}$ is path-homotopic to the constant path at $f(0)$.