MA 571 Qualifying Exam. January 1996.

Each problem is worth 12 points. There are 96 points altogether.

1. Let X be a metric space with metric d. Prove that d is continuous as a function from $X \times X$ to $\mathbb{R}$, where $X \times X$ has the "product metric" d' defined by
$$d'((x,y),(x_1,y_1)) = \max\left(d(x,x_1), d(y,y_1)\right)$$
(you may assume d' is indeed a metric, you do not have to prove it).

2. Let X be an ordered set with the order topology. If X is connected, prove that X has the least upper bound property.

3. Let X be the closed unit disk in $\mathbb{R}^2$, and define an equivalence relation on X by: $a \sim b$ if either $a = b$ or a and b are both on the unit circle. Let X^* be the partition of X into the equivalence classes of $\sim$. Show that X^*, with its quotient topology, is homeomorphic to the one-point compactification of the open unit disk.

4. Show that every locally compact Hausdorff space is regular.

5. A *manifold* is a Hausdorff space with the property that every point has a neighborhood homeomorphic to $\mathbb{R}^n$. Prove that every manifold is regular.

6. Prove that every compact Hausdorff space is a Baire space.

7. Let $p : E \to B$ be a covering map with B connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. Prove that, for every $b \in B$, $p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

8. Let X be a topological space and $f : [0,1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1-t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.