1. Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a function with the property that

$$f(A) \subseteq f(A)$$

for all subsets $A$ of $X$.

**Prove** that $f$ is continuous.

2. Let $X$ be a compact space and suppose we are given a nested sequence of subsets

$$C_1 \supset C_2 \supset \cdots$$

with all $C_i$ closed. Let $U$ be an open set containing $\bigcap C_i$.

**Prove** that there is an $i_0$ with $C_{i_0} \subseteq U$.

3. Prove the Tube Lemma: let $X$ and $Y$ be topological spaces with $Y$ compact, let $x_0 \in X$, and let $N$ be an open set of $X \times Y$ containing $\{x_0\} \times Y$, then there is an open set $W$ of $X$ containing $x_0$ with $W \times Y \subseteq N$.

4. Let $f : X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let $U$ be an open set containing $f^{-1}(y)$.

**Prove** that there is an open set $V$ containing $y$ such that $f^{-1}(V)$ is contained in $U$.

5. Let $p : \mathbb{R} \to S^1$ be the usual covering map (specifically, $p(t) = (\cos 2\pi t, \sin 2\pi t)$). Let $b_0 \in S^1$ be the point $(1,0)$. Recall that the standard map

$$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is a lifting of $f$ with $\tilde{f}(0) = 0$.

(a) (14 points) **Prove** that $\phi$ is 1-1.

(b) (14 points) **Prove** that $\phi$ is a group homomorphism.
6. Let $X$ be a topological space and let $x_0 \in X$.

Let $U$ and $V$ be open sets containing $x_0$, and suppose that the hypotheses of the Seifert-van Kampen theorem are satisfied (that is, $U \cup V = X,$

$$U \cap V = \emptyset,$$

and $U, V, U \cap V$ are path-connected).

Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps.

Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto.

Prove, using the Seifert-van Kampen theorem, that $(j_2)_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is onto.