MA 571 Qualifying Exam. January 2007. J. McClure, Y-J Lee

Each problem is worth 14 points and you get two points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

1. **Prove** that the countable product

$$\prod_{n=1}^{\infty} \mathbb{R}$$

(with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point

$$\mathbf{0} = (0, 0, \dots)$$

such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

Note: the book proves that $\prod_{n=1}^{\infty} \mathbb{R}$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

2. Given:

- $p : X \to Y$ is a quotient map.
- Y is connected.
- For every $y \in Y$, the set $p^{-1}(\{y\})$ is connected.

Prove that X is connected.

3. Let S^1 denote the circle

$$\{x^2 + y^2 = 1\}$$

in $\mathbb{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do *not* have to prove that this is an equivalence relation). **Prove** that the quotient space $S^1/\sim$ is homeomorphic to S^1.

4. Let A be a subset of $\mathbb{R}^2$ which is homeomorphic to the closed unit interval I.

Prove that A does not contain a nonempty open set.

5. Let X be a path-connected space.

   Let x_0 and x_1 be two different points in X.

   Suppose that every path from x_0 to x_1 is path-homotopic to every other path from x_0 to x_1.

   Prove that X is simply-connected.

6. Let $p : E \to B$ be a covering map. **Prove** that p takes open sets to open sets.

7. Let a and b denote the points $(-1, 0)$ and $(1, 0)$ in $\mathbb{R}^2$. Let z denote the origin $(0, 0)$.

   Prove, using the Seifert-van Kampen theorem, that $\pi_1(\mathbb{R}^2 - \{a, b\}, z)$ is a free group on two generators. You may not use any other method.

   If you use deformation retractions, you should write down the retraction map r in each case but you don't have to write down the homotopy H.