MA 571 Qualifying Exam. January 2009. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous. Define $h : X \to \mathbb{R}$ by
$$h(x) = \min\{f(x), g(x)\}$$
Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

2. Let X be a topological space and A a subset of X. Suppose that
$$A \subset \overline{X - \overline{A}}.$$
Prove that $\overline{A}$ does not contain any nonempty open set.

3. Let X be a compact Hausdorff space and let $f : X \to X$ be a continuous function. Suppose f is 1-1. **Prove** that there is a nonempty closed set A with $f(A) = A$.

4. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. **Prove** that the quotient space $\mathbb{R}^2/\sim$ is compact but not Hausdorff.

5. Let X be a topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives a homomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this).

   Suppose that for every pair of paths α and β from x_0 to x_1 the homomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. **Prove** that $\pi_1(X, x_0)$ is abelian.

6. Let $p : E \to B$ be a covering map with B connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. **Prove** that, for every $b \in B$, $p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

7. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

   i) Calculate $H_1(X)$. (You may use any fact in Munkres, but be sure to be clear about what you're using.)

   ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?