MA 571 Qualifying Exam. August 2013 Professor R. Kaufmann

INSTRUCTIONS
There are 7 problems, two of them are on the back. Each problem is worth 10 points.
Unless otherwise stated, you may use anything in Munkres’s book—but be care-ful to make it clear what fact you are using.
When you use a set theoretic fact that isn’t obvious, be careful to give a clear explanation. To disprove a statement, give a counterexample and prove that it has the necessary properties.

PROBLEMS

1. (a) Give the definition of a sequence of points $x_n$ converging to a point $x$ in a topological space $X$.
   (b) Prove or disprove! A function $f : X \rightarrow Y$ is continuous if and only if for all sequences $x_n$ converging to a point $x$ in $X$ the points $f(x_n)$ converge to $f(x)$.

2. For an uncountable product $\mathbb{R}^J$ in the product topology let $A$ be the subspace consisting of all points $(x_\alpha)$, for which $x_\alpha = 1$ for all but finitely many $\alpha \in J$. Show that there is a point in the closure of $A$ which is not the limit of a sequence in $A$. Conclude using the Sequence Lemma that an uncountable product $\mathbb{R}^J$ in the product topology is not metrizable.

3. If $f$ and $g$ are continuous functions on a topological space $X$ with values in a Hausdorff space $Y$ and $f$ and $g$ agree on a dense subset of $X$, then $f = g$.

4. Let $X$ be a locally compact Hausdorff space, let $Y$ be any space, and let the function space $\mathcal{C}(X,Y)$ have the compact-open topology. Prove that the map
   
   $e : X \times \mathcal{C}(X,Y) \rightarrow Y$

   defined by the equation
   
   $e(x, f) = f(x)$

   is continuous.

5. Let $X$ be the subspace $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, and let $f : X \rightarrow \mathbb{R}$ be the function defined by $f(x, y) = x$. Prove or disprove!
   (a) $f$ is a continuous map.
   (b) $f$ is an open map.
   (c) $f$ is a closed map.
   (d) $f$ is a quotient map.
6. Let $X$ be the space obtained by attaching two discs to $S^1$, where the first disc $D_1$ is attached via the map $\partial D_1 = S^1 \to S^1, z \to z^3$ and the second disc is attached by $g : \partial D_2 = S^1 \to S^1 : z \to z^5$. Compute $\pi_1(X)$ and $H_1(X)$. (Hint use the special case of Seifert–van Kampen for adjoining 2–cells).

7. The Klein bottle $K$ is the space obtained from a square by the labelling scheme $aba^{-1}b$.

   (a) Give a presentation of the fundamental group of $S = K \# K$, that is the connected sum of two copies of $K$.

   (b) Give the standard surface in the classification of surfaces that $S$ is homeomorphic to and prove your answer.