MA 571 Qualifying Exam. January 2013. Professor R. Kaufmann

Instructions

There are 8 problems, two of them are on the back. Each problem is worth 12 points and you get four points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

Problems

1. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that
$$f(\overline{A}) \subset \overline{f(A)}$$
for all subsets A of X.

   Prove that f is continuous.

2. Let $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ and in general let C_i be given by removing the (open) middle third of each intervals making up C_{i-1}. Let $K = \bigcap_{i=1}^{\infty} C_i$ considered as a subspace of $[0, 1]$. This is called the Cantor set. This set is equal to the set of $x \in [0, 1]$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$ for all n.

   Prove that

   (a) K is closed.

   (b) Every point of K is a limit point of K

   (c) K is not homeomorphic to a closed interval.

3. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{ (x, f(x)) \mid x \in X \}$ of $X \times Y$.

   Prove that if Y is Hausdorff then G_f is closed.

4. Show that if $\displaystyle\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

5. Let X be a locally connected space and let $f : X \to Y$ be a quotient map. Show that Y is also locally connected. (Hint: If C is a component of an open set U of Y, then $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.)

6. Let $p : Y \to X$ be a covering map, let $y \in Y$, and let $x = p(y)$.

   Let σ be a loop beginning and ending at x and let $[\sigma]$ be the corresponding element of $\pi_1(X, x)$.

   Let $\tilde{\sigma}$ be the unique lifting of σ to a path starting at y.

   Prove that if $[\sigma] \in p_*\pi_1(Y, y)$ then $\tilde{\sigma}$ ends at y.

7. Consider the torus $T = S^1 \times S^1$ as $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts by translation. Let $\sim$ be the equivalence relation given by $(x, y) \sim (-x, -y)$. Let $X := T/\sim$ be the quotient space.

   (a) **Prove or disprove:** The quotient map $p : T \to X$ is a covering map.

   (b) **Prove or disprove:** X is homeomorphic to a quotient space obtained from a pasting scheme.

   (c) **Prove or disprove:** X is a surface (topological 2–manifold).

   (d) **Prove or disprove:** $\pi_1(X)$ is trivial.

8. Let $aba^{-1}ba$ be the labeling scheme of a pentagon.

   (a) Show that the quotient space corresponding to the labeling is homeomorphic to the connected sum $D_3 \# P^2$ of the 3–fold dunce cap and the real projective plane.

   (b) Calculate π_1 and H_1 of the resulting space (for H_1 write the result in standard form for finitely presented Abelian groups, i.e. as a free Abelian group times finite cyclic groups).