Exam 1, Math 162, Spring 2014, Version 02

#1. Area $= |\vec{a} \times \vec{b}|$.

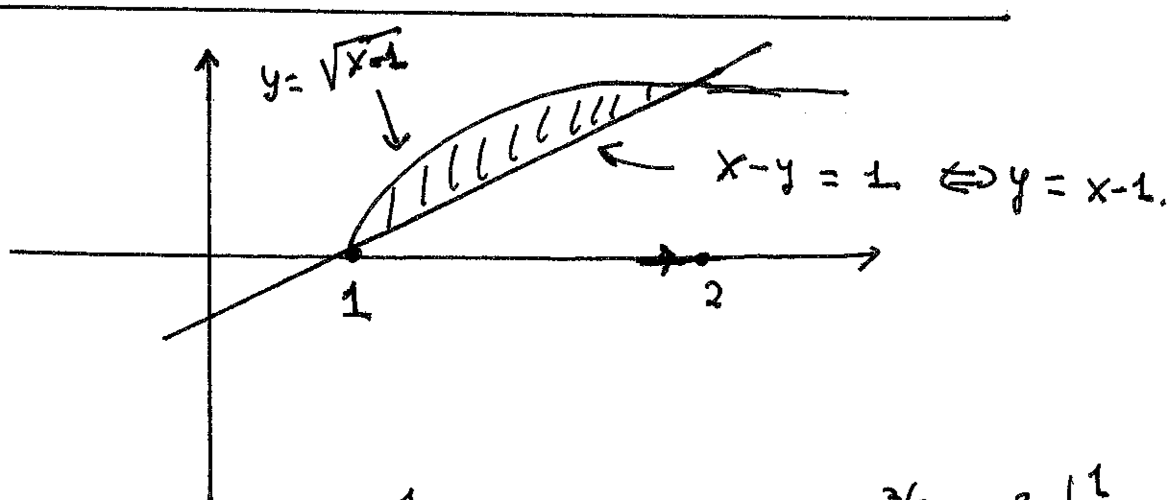
Step 1: $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ -2 & 3 & -1 \end{vmatrix} = i \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} - j \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} + k \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}$

$= i(2-3) - j(-3+2) + k(9-4)$

$= -i + j + 5k$.

Step 2: $|\vec{a} \times \vec{b}| = \sqrt{1^2+1^2+5^2} = \sqrt{27}$. C

#2.



Area $= \displaystyle\int_1^2 \left[\sqrt{x-1} - (x-1)\right] dx = \int_0^1 (\sqrt{u} - u)\, du = \frac{2}{3}u^{3/2} - \frac{u^2}{2}\Big|_0^1$

$u = x-1$, $du = dx$

$= \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$ C

#3. $x^2 + y^2 + z^2 - 6x + 2y - 4z + 13 = (x-3)^2 - 9 + (y+1)^2 - 1 + (z-2)^2 - 4 + 13 = 0$

$\Rightarrow (x-3)^2 + (y+1)^2 + (z-2)^2 = 14 - 13 = 1$.

Distance from $(3, -1, 2)$ to $(1, 0, 4)$

$= \sqrt{(3-1)^2 + 1^2 + (4-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$. $\boxed{B}$

#4) Given: (i) $|\vec{a}| = 1$, (ii) $\vec{a} \cdot \vec{b} = 0$.

(i) $\Leftrightarrow \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + a^2} = 1 \Leftrightarrow \sqrt{\frac{1}{4} + \frac{1}{16} + a^2} = 1 \Leftrightarrow \sqrt{a^2 + \frac{5}{16}} = 1$

$\Leftrightarrow a^2 + \frac{5}{16} = 1 \Leftrightarrow a^2 = \frac{11}{16} \Leftrightarrow a = \pm \frac{\sqrt{11}}{4}$.

(ii) $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \frac{1}{2} b + 0 \cdot \frac{1}{4} + 2a = 0 \Leftrightarrow b = -4a. \Leftrightarrow$

$b = \pm\sqrt{11}$. $\boxed{D}$

#5. $\int_0^{\pi/4} \sec^4 x \tan^2 x \, dx = \int_0^{\pi/4} \sec^2 x \tan^2 x \underbrace{\sec^2 dx}$

$= \int_0^{\pi/4} (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx$ $\boxed{\text{Use } \tan^2 x + 1 = \sec^2 x}$

$u = \tan x, \quad du = \sec^2 x \, dx.$

$= \int_0^1 (u^2+1) u^2 \, du = \int_0^1 (u^4 + u^2) du = \frac{u^5}{5} + \frac{u^3}{3} \Big|_0^1 = \frac{1}{5} + \frac{1}{3} = \boxed{\frac{8}{15}}$

$\boxed{B}$

#6. $\int_1^2 x \ln(x) dx = \frac{x^2}{2} \ln(x) \Big|_1^2 - \frac{1}{2} \int_1^2 x \, dx = \frac{x^2 \ln(x)}{2} - \frac{1}{4} x^2 \Big|_1^2$

$\boxed{\begin{array}{ll} u = \ln x, & dv = x\,dx \\ du = \frac{1}{x} dx, & v = \frac{x^2}{2} \end{array}}$

$= 2\ln(2) - \left[\frac{1}{4} \cdot 4 - \frac{1}{4}\right]$

$= \boxed{2\ln(2) - \frac{3}{4}}$ $\boxed{B}$

#7) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-4x^2}} = \int_0^{1/2} \frac{dx}{\sqrt{4(\frac{1}{4}-x^2)}} = \frac{1}{2}\int_0^{1/2} \frac{dx}{\sqrt{\frac{1}{4}-x^2}}$

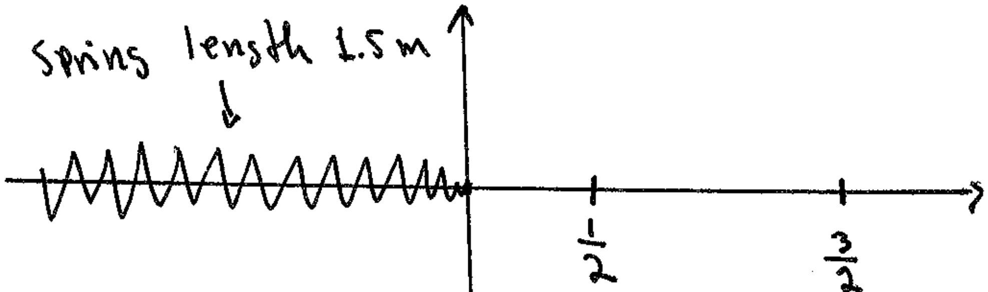
$x = \frac{1}{2}\sin\theta, \quad dx = \frac{1}{2}\cos\theta\, d\theta, \quad \sqrt{\frac{1}{4}-x^2} = \sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta} = \frac{1}{2}\cos\theta$

When $x = 0, \ \theta = 0, \quad x = \frac{1}{2}, \ \theta = \pi/2.$

$\displaystyle = \frac{1}{4}\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\frac{1}{2}\cos\theta} = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\,\theta\Big|_0^{\pi/2} = \boxed{\frac{\pi}{4}}$

$\boxed{C}$

#8. Hooke's Law: Force $f(x) = kx$. Need k first.



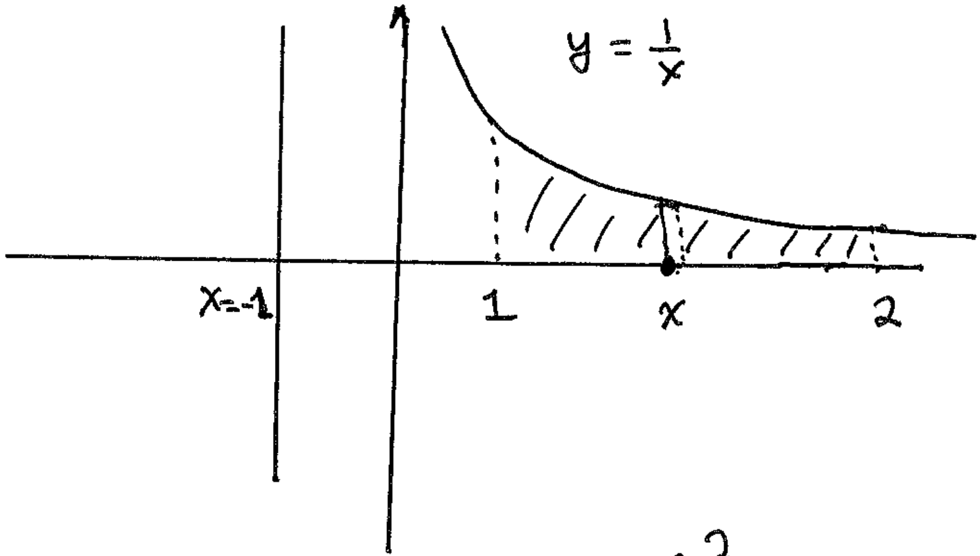
$\displaystyle \frac{1}{4} = \int_0^{1/2} kx\,dx = \frac{kx^2}{2}\Big|_0^{1/2} \Rightarrow k = 2.$

$= \frac{k}{8}$

$\displaystyle W = 2\int_{1/2}^{3/2} x\,dx = x^2\Big|_{1/2}^{3/2} = \cancel{\frac{9}{4} - \frac{8}{4}} = \frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2\text{J}.$

$\boxed{E}$

#9



$$\text{Area} = 2\pi \int_1^2 (x+1)\frac{1}{x}\,dx = 2\pi \int_1^2 \left(1 + \frac{1}{x}\right) dx$$

$$= 2\pi \left[x + \ln(x) \Big|_1^2 \right] = 2\pi \left[2 + \ln(2) - 1 \right]$$

$$= 2\pi \left[1 + \ln(2) \right]. \quad \boxed{D}$$

#10. $$\int \sqrt{9x^2 - 1}\,dx = \int \sqrt{9\left(x^2 - \frac{1}{9}\right)}\,dx = 3 \int \sqrt{x^2 - \frac{1}{9}}\,dx$$

$$x = \frac{1}{3}\sec\theta, \quad dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$$

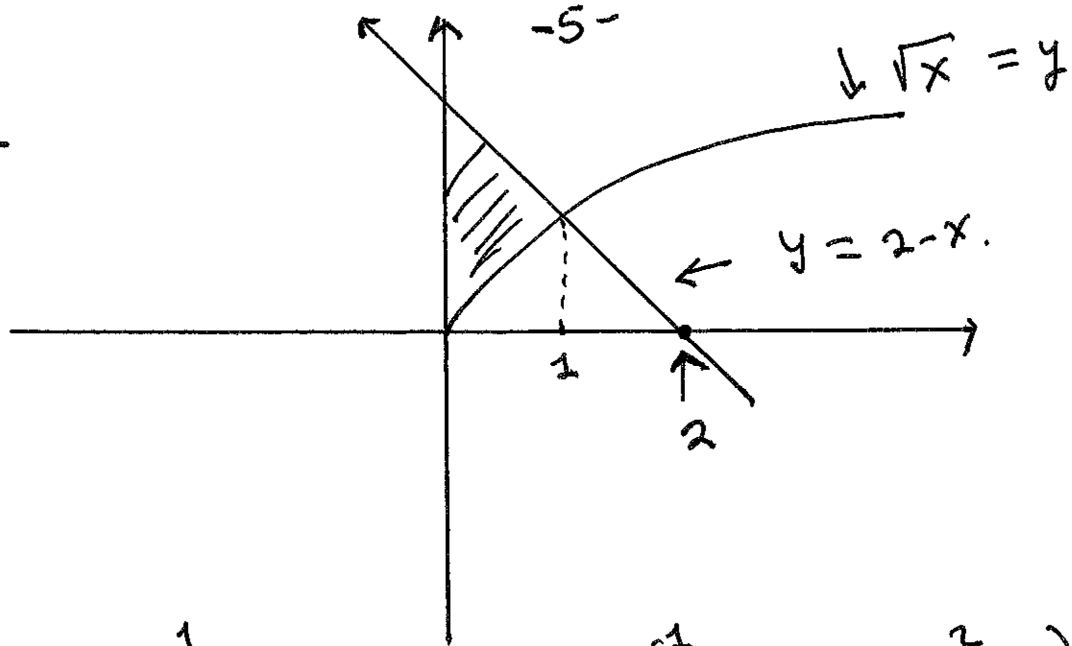
$$\sqrt{x^2 - \frac{1}{9}} = \sqrt{\frac{1}{9}\sec^2\theta - \frac{1}{9}} = \frac{1}{3}\sqrt{\sec^2 - 1}$$

$$= \frac{1}{3}\tan\theta.$$

$$= 3\int \frac{1}{3}\tan\theta \cdot \frac{1}{3}\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \tan^2(\theta)\sec(\theta)\,d\theta$$

$$\boxed{E}$$

#11

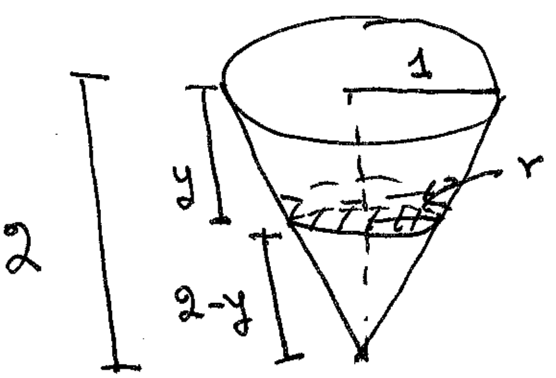


$$\text{Area} = \pi \int_0^1 \left[(2-x)^2 - x\right] dx = \pi \int_0^1 (4 - 4x + x^2 - x)\, dx$$

$$= \pi \int_0^1 (4 - 5x + x^2)\, dx = \pi \left[4x - \frac{5}{2}x^2 + \frac{x^3}{3} \Big|_0^1 \right] = \pi \left[\frac{1}{3} - \frac{5}{2} + \frac{8}{2} \right]$$

$$= \boxed{\frac{11}{6}\pi} \qquad \boxed{E}$$

#12.



$$\frac{r}{2-y} = \frac{1}{2}, \quad r = \frac{1}{2}(2-y).$$

Volume of cross section

$$= \pi\left[\frac{1}{2}(2-y)\right]^2 \cdot \Delta y.$$

$$= \frac{\pi}{4}(2-y)^2 \Delta y.$$

$$W = \frac{\pi}{4} \rho \int_0^2 (2-y)^2 \underbrace{y}_{\text{distance}}\, \underbrace{dy}_{\Delta y} = \frac{\pi \rho}{4} \int_0^2 (4 - 4y + y^2)\, y\, dy$$

$$= \frac{\pi \rho}{4} \int_0^2 (y^3 - 4y^2 + 4y)\, dy = \frac{\rho \pi}{4} \left[\frac{y^4}{4} - \frac{4}{3}y^3 + 2y^2 \Big|_0^2 \right] \qquad \boxed{A}$$

$$= \frac{\rho \pi}{4} \left[\frac{16}{4} - \frac{4 \cdot 8}{3} + 2 \cdot 4 \right] = \rho \pi \left[1 - \frac{8}{3} + 2 \right] = \rho \pi \left[3 - \frac{8}{3} \right] = \boxed{\frac{\rho \pi}{3}}$$