The following formulas were given on the exam:

Moments and center of mass

\[ M_x = \int_a^b \frac{1}{2} \left( (f(x))^2 - (g(x))^2 \right) \, dx, \quad M_y = \int_a^b x (f(x) - g(x)) \, dx \]

\[ \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}, \]

Arc length

\[ L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \]

Area of a surface of revolution

\[ S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

1) The mass of the region bounded by \( f(x) = \frac{1}{2} \sqrt{4 - 2x^2} \), \( g(x) = -\frac{1}{2} \sqrt{4 - 2x^2} \) and the \( y \)-axis is \( \pi \sqrt{2} \). Its center of mass is

A) \( (\frac{8}{3\pi\sqrt{2}}, 0) \)

B) \( (0, \frac{8}{3\pi\sqrt{2}}) \)

C) \( (\frac{1}{2}, \frac{8}{3\pi\sqrt{2}}) \)

D) \( (0, \frac{4}{3\pi\sqrt{2}}) \)

E) \( (\frac{4}{3\pi\sqrt{2}}, 0) \)

Solution: According to the formulas above, the center of mass is

\[ \bar{x} = \frac{1}{\pi\sqrt{2}} \int_0^{\sqrt{2}} x \sqrt{4 - 2x^2} \, dx, \]

\[ \bar{y} = 0 \quad \text{this is because} \quad f(x)^2 = g(x)^2. \]
To compute this integral we just set $u = 4 - 2x^2$. Then $du = -4x \, dx$ and the integral becomes

$$I = \frac{1}{\pi \sqrt{2}} \int_0^4 \sqrt{u} \, du = \frac{1}{\pi \sqrt{2}} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_0^4 = \frac{1}{\pi \sqrt{2}} \left( \frac{8}{3} - 0 \right) = \frac{8}{3\pi \sqrt{2}}.$$

The correct answer is E.

2) The improper integral

$$\int_0^1 \ln x \, dx = \lim_{a \to 0} \int_a^1 \ln x \, dx.$$

A) $2 \ln 2$

B) $-4 \ln 2$

C) $2 \ln 2$

D) $-1$

E) $-\frac{1}{9}$

Solution: By definition of improper integrals

$$\int_0^1 \ln x \, dx = \lim_{a \to 0} \int_a^1 \ln x \, dx.$$

Integration by parts gives

$$\int_a^1 \ln x \, dx = (x \ln x - x)|_a^1 = -1 - (a \ln a - a).$$

Now we have to compute

$$\lim_{a \to 0} a \ln a - a = \lim_{a \to 0} a \ln a.$$

To do this we use L’Hopital’s rule and write

$$\lim_{a \to 0} a \ln a = \lim_{a \to 0} \frac{\ln a}{\frac{1}{a}} = \lim_{a \to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a \to 0} -a = 0.$$

So

$$\int_0^1 \ln x \, dx = -1.$$

The correct answer is D. Unfortunately this question had two identical alternatives, but both are incorrect.
3) The improper integral
\[ \int_0^\infty x e^{-x^2} \, dx \] is equal to

A) \( \frac{1}{3} \)
B) \( \frac{1}{4} \)
C) \( \frac{1}{2} \)
D) 1
E) 2

Solution: By definition
\[ \int_0^\infty x e^{-x^2} \, dx = \lim_{M \to \infty} \int_0^M x e^{-x^2} \, dx. \]
To compute the integral we set \( y = x^2 \) and so \( dy = 2x \, dx \). Therefore
\[ \int_0^M x e^{-x^2} \, dx = \frac{1}{2} \int_0^{M^2} e^{-y} \, dy = \frac{1}{2} \left( 1 - e^{-M^2} \right). \]
So
\[ \int_0^\infty x e^{-x^2} \, dx = \lim_{M \to \infty} \int_0^M x e^{-x^2} \, dx = \lim_{M \to \infty} \frac{1}{2} \left( 1 - e^{-M^2} \right) = \frac{1}{2}. \]
The correct answer is C.

4) The length of the curve \( y = \frac{x^3}{6} + \frac{1}{2x}, \ 1 \leq x \leq 2 \) is

A) \( \frac{17}{12} \)
B) \( \frac{4}{3} \)
C) \( \frac{3}{2} \)
D) 2
E) 1

Solution: The derivative of \( f(x) = \frac{x^3}{6} + \frac{1}{2x} \) is \( f'(x) = \frac{x^2}{2} - \frac{1}{2x^2} \). So according to the
formula given on the first page, the length of the curve is

\[ L = \int_1^2 \sqrt{1 + \left( \frac{x^2}{2} - \frac{1}{2x^2} \right)^2} \, dx \]

Notice that

\[ 1 + \left( \frac{x^2}{2} - \frac{1}{2x^2} \right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left( \frac{x^2}{2} + \frac{1}{2x^2} \right)^2. \]

Therefore

\[ L = \int_1^2 \left( \frac{x^2}{2} + \frac{1}{2x^2} \right) \, dx = \left. \left( \frac{x^3}{6} - \frac{1}{2x} \right) \right|_1^2 = \left( \frac{8}{6} - \frac{1}{4} \right) - \left( \frac{1}{6} - \frac{1}{2} \right) = \frac{17}{12}. \]

The correct answer is A.

5) The area of the surface obtained by rotating the curve

\[ y = x^3, \quad 0 \leq x \leq 1 \]

about the x-axis is

A) \( \pi \sqrt{3} \)

B) \( \frac{2\pi}{3} \)

C) \( \frac{\pi}{27} (10\sqrt{10} - 1) \)

D) \( 6\pi(3\sqrt{3} - 1) \)

E) \( \frac{\pi}{3} (10\sqrt{10} - 1) \)

Solution: The derivative of \( f(x) = x^3 \) is \( f'(x) = 3x^2 \) so according to the formula on

\[ A = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} \, dx. \]

Set \( u = 1 + 9x^4 \). Then \( du = 36x^3 \, dx \) and the integral becomes:

\[ A = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} \, dx = \frac{2\pi}{36} \int_1^{10} u^{\frac{3}{4}} \, du = \frac{2\pi}{36} \frac{2}{3} u^{\frac{7}{4}} \bigg|_1^{10} = \frac{2\pi}{36} \frac{2}{3} (10\sqrt{10} - 1) = \frac{\pi}{27} (10\sqrt{10} - 1). \]

The correct answer is C.
6) Find
\[
\lim_{n \to \infty} \frac{\sqrt{2n^4 + n^2 + 1}}{2n^2 + 1}
\]
A) \(\frac{1}{2}\)
B) 1
C) \(\frac{\sqrt{2}}{2}\)
D) 0
E) It does not exist.

Solution: We just write
\[
\lim_{n \to \infty} \frac{\sqrt{2n^4 + n^2 + 1}}{2n^2 + 1} = \lim_{n \to \infty} \frac{n^2(2 + \frac{1}{n^2} + \frac{1}{n^4})}{n^2(2 + \frac{1}{n^2})} = \lim_{n \to \infty} \frac{\sqrt{2 + \frac{1}{n^2} + \frac{1}{n^4}}}{2 + \frac{1}{n^2}} = \frac{\sqrt{2}}{2}.
\]
The correct answer is C.

7) The series
\[
\sum_{n=1}^{\infty} \frac{\sqrt{2n^4 + n^2 + 1}}{2n^2 + 1}
\]
A) diverges
B) converges conditionally
C) converges by the ratio test
D) converges by the root test
E) converges by the integral test

Solution: We just computed in question 6 that \(\lim_{n \to \infty} \frac{\sqrt{2n^4 + n^2 + 1}}{2n^2 + 1} = \frac{\sqrt{2}}{2}\). Since this limit is not equal to zero, the series diverges. The correct answer is A.
8) Find
\[ \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) \]
A) 0 
B) 1 
C) 2 
D) \infty 
E) it does not exist 

Solution: We know that 
\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]
We just the write
\[ \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) = \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} \]
Set \( \frac{1}{n} = x \). So when \( n \to \infty, \ x \to 0 \). Therefore
\[ \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) = \lim_{x \to 0} \frac{\sin x}{x} = 1. \]
The correct answer is B.

9) The series \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \)
A) converges by the ratio test 
B) diverges by the ratio test 
C) converges because \( \lim_{n \to \infty} \sin \left( \frac{1}{n} \right) = 0 \)
D) converges by the limit comparison test with \( \sum_{n=1}^{\infty} \frac{1}{n} \)
E) diverges by the limit comparison test with \( \sum_{n=1}^{\infty} \frac{1}{n} \)

Solution: The limit comparison theorem states that if \( \sum a_n \) and \( \sum b_n \) are series of positive terms and and \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \), with \( L \neq 0 \) and \( L \neq \infty \), then the series \( \sum a_n \) and \( \sum b_n \)
either converge or diverge simultaneously. That is one cannot converge and the other diverge.

In problem 8 we found that
\[ \lim_{n \to \infty} n \sin \frac{1}{n} = \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1. \]

Since the series \( \sum \frac{1}{n} \) diverges, \( \sum \sin \left( \frac{1}{n} \right) \) must diverge as well. The correct answer is E.

We remark that this is problem number 31 of lesson 20, which was assigned in the homework.

10) \[ \sum_{n=0}^{\infty} \frac{2^{n+1} - 3^n}{6^n} = \]
A) 1
B) 5
C) \( \frac{7}{3} \)
D) \( \frac{1}{2} \)
E) diverges

Solution: Recall that if \(|r| < 1\) then
\[ \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \]
So we write
\[ \sum_{n=0}^{\infty} \frac{2^{n+1} - 3^n}{6^n} = \sum_{n=0}^{\infty} \frac{2^n}{6^n} - \frac{3^n}{6^n} = 2 \sum_{n=0}^{\infty} \frac{1}{3^n} - \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \frac{1}{1 - \frac{1}{3}} - \frac{1}{1 - \frac{1}{2}} = 3 - 2 = 1. \]
The correct answer is A.

11) What is the smallest number of terms of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \) that need to be added to compute its sum with error strictly less than \( 10^{-2} \)?
A) 3
B) 4
C) 5

D) 6

E) 7

Solution: Recall that for a converging alternating series \( \sum_{n=0}^{\infty}(-1)^n b_n \) the difference between the sum of the series and the sum of the first \( N \) terms satisfies:

\[
\left| \sum_{n=0}^{\infty}(-1)^n b_n - \sum_{n=0}^{N}(-1)^n b_n \right| \leq b_{N+1}.
\]

In our case \( b_n = \frac{1}{n!} \) and therefore we want \( N \) such that \( b_{N+1} < 10^{-2} \). That is

\[
\frac{1}{(N+1)!} < \frac{1}{100}
\]

Hence we want the first \( N \) such that

\[
(N+1)! > 100
\]

That is \( N = 4 \). So the correct answer is B.

12) Which of the following is a correct statement about the series

\[
S_1 = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad \text{and} \quad S_2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}?
\]

A) \( S_1 \) and \( S_2 \) are divergent

B) \( S_1 \) converges but \( S_2 \) diverges

C) \( S_1 \) diverges but \( S_2 \) converges conditionally

D) \( S_1 \) converges and \( S_2 \) converges conditionally

E) \( S_1 \) and \( S_2 \) converge absolutely

Solution: By the integral test \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converges if and only if the integral \( \int_{2}^{\infty} \frac{1}{x \ln x} \, dx \) converges. To compute this integral we set \( \ln x = u \). Then \( du = \frac{1}{x} \, dx \) and hence

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{M \to \infty} \int_{\ln 2}^{\ln M} \frac{1}{u} \, du = \lim_{M \to \infty} (\ln(\ln M) - \ln(\ln 2)) = \infty.
\]

As the integral diverges, so does the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \).

On the other hand \( b_n = \frac{1}{n \ln n} \) satisfies

i) \( b_n \geq 0 \)
ii) $b_{n+1} \leq b_n$

iii) $\lim_{n \to \infty} b_n = 0.$

Thus by the alternating series test $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ converges conditionally. The correct answer is C.

13) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}.$$  

A) $(-3, 3)$

B) $\left(-\frac{1}{3}, \frac{1}{3}\right)$

C) $(-3, 3)$

D) $\left(-\frac{1}{3}, \frac{1}{3}\right)$

E) $[-3, 3)$

Solution: First we use the ratio test to find the radius of convergence: It says that if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

and $L < 1$, then $\sum |a_n|$ converges. In this case $a_n = \frac{x^n}{n 3^n}$. So

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n 3^n}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{3} \cdot \frac{n}{n+1} \right| = \frac{|x|}{3}.$$ 

So the series converges if $\left| \frac{x}{3} \right| < 1$. That is it converges in $(-3, 3)$. Now we need to test the end points of this interval. When $x = 3$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

and this diverges. When $x = -3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

which converges. So the series converges for $x$ in $[-3, 3)$. The correct answer is E.