MA162 — EXAM III — SPRING 2017 — APRIL 11, 2017
TEST NUMBER 01

INSTRUCTIONS:

1. **Do not open the exam booklet until you are instructed to do so.**
2. **Before you open the booklet fill in the information below and use a # 2 pencil to fill in the required information on the scantron.**
3. MARK YOUR TEST NUMBER ON YOUR SCANTRON
4. Once you are allowed to open the exam, make sure you have a complete test. There are 8 different test pages (including this cover page).
5. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers on this test booklet.
6. There are 14 problems and the number of points each problem is worth is indicated next to the problem number. The maximum possible score is 100 points. No partial credit.
7. Do not leave the exam room during the first 20 minutes of the exam.
8. If you do not finish your exam in the first 50 minutes, you must wait until the end of the exam period to leave the room.
9. After you have finished the exam, hand in your scantron and your test booklet to your recitation instructor.

DON'T BE A CHEATER:

1. Do not give, seek or obtain any kind of help from anyone to answer questions on this exam. If you have doubts, consult only your instructor.
2. Do not look at the exam or scantron of another student.
3. Do not allow other students to look at your exam or your scantron.
4. You may not compare answers with anyone else or consult another student until after you have finished your exam, given it to your instructor and left the room.
5. Do not consult notes or books.
6. **Do not handle** phones or cameras, calculators or any electronic device until after you have finished your exam, given it to your instructor and left the room.
7. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs collect the scantrons and the exams.
8. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above statements regarding academic dishonesty:

STUDENT NAME: SOLUTIONS

STUDENT SIGNATURE: ______________________

STUDENT ID NUMBER: ______________________

SECTION NUMBER AND RECITATION INSTRUCTOR: ______________________

FORMULA SHEET

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{for all } x.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x.$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \text{ for all } x.$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}x^n \dots$$
$$\text{for } |x| < 1.$$

If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$, for $|x-a| < R$, then $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1}$ for $|x-a| < R$.

If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$, for $|x-a| < R$, then $f'(x)\,dt = \sum_{n=1}^{\infty} nC_n (x-a)^{n-1}$ for $|x-a| < R$.

1. (8 points) Consider the two series

$$I)\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+7} \text{ and } II)\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{5^n}.$$

$\frac{n}{n^2+7} \sim 1/n$ for n large.

$\sum_{n=1}^{\infty} \frac{n}{n^2+7}$ diverges.

Which of the following is true?

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+7}$ is alternating so it converges.

A. I and II converge conditionally

B. I converges conditionally and II converges absolutely

C. I converges absolutely and II converges conditionally

(I) converges conditionally

D. I and II converge absolutely

E. I and II diverge

In (II) $a_n = (-1)^n \frac{n}{5^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n} = \frac{n+1}{n} \cdot \frac{1}{5} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1$$

II converges absolutely

2. (8 points) The series $\sum_{n=1}^{\infty}\frac{1}{n(\ln(2n))^2}$

The Integral test gives that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p>1$. Here $p=2$.

A. Converges by the ratio test

B. Diverges by the ratio test

C. Diverges by the integral test

D. Converges by the integral test

E. Converges by the root test.

The ratio test and the root test are inconclusive

3. (8 points) Which of the following series are convergent?

I. $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$

II. $\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^2}$

III. $\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^{4/3}}$

$\frac{3n^2}{n^3+1} \sim 3/n$ for n large. Series diverges

$\frac{3n^2}{(n^3+1)^2} \sim \frac{3n^2}{n^6} = 3/n^4$ for n large. Converges

$\frac{3n^2}{(n^3+1)^{4/3}} \sim \frac{3n^2}{n^4} = 3/n^2$. Converges.

A. II only

B. I and II only

C. II and III only

D. None of them

E. All of them

4. (8 points) For what values of a is the series $\sum_{n=1}^{\infty} \left(\frac{n}{an+1}\right)^n$ absolutely convergent?

A. $|a| \le 1$

B. $|a| < 1$

C. $|a| \ge 1$

D. $|a| > 1$

E. $|a| = 1$

$a_n = \left(\frac{n}{an+1}\right)^n$. $|a_n|^{1/n} = \frac{n}{|an+1|}$.

$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{|a|} < 1$.

$\boxed{|a| > 1}$

The case $a = 1$ $\quad a_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e}$ as $n \to \infty$.

$a = -1$. $\quad a_n = \left(\frac{n}{1-n}\right)^n = (-1)^n \frac{1}{(1-1/n)^n}$

$\lim_{n\to\infty} a_n$ does not exist

5. (8 points) Compute the Taylor series of $f(x) = \ln x$ centered at 4 and use it to find $f^{(9)}(4)$.

A. $f^{(9)}(4) = \frac{9!}{4^9}$

B. $f^{(9)}(4) = -\frac{9!}{4^9}$

C. $\boxed{f^{(9)}(4) = \frac{8!}{4^9}}$

D. $f^{(9)}(4) = -\frac{8!}{4^9}$

E. $f^{(9)}(4) = \frac{10!}{4^9}$

$$\frac{1}{x} = \frac{1}{4+x-4} = \frac{1}{4}\,\frac{1}{1+\frac{x-4}{4}}$$

Recall that $\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n$, for $|r|<1$

$$\therefore \frac{1}{1+\frac{x-4}{4}} = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{4^n}, \quad \left|\frac{x-4}{4}\right| < 1.$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{4^{n+1}}.$$

$$\ln x - \ln 4 = \int_4^x \frac{1}{t}\,dt = \sum_{n=0}^{\infty} \int_4^x (-1)^n \frac{(t-4)^n}{4^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^{n+1}}{(n+1)4^{n+1}}.$$

When $\underline{n+1 = 9}$ $\quad \frac{f^{(9)}(4)}{9!} = (-1)^8 \frac{1}{9 \cdot 4^9}$; $f^{(9)}(4) = \frac{9!}{9\,4^9} = \frac{8!}{4^9}$

6. (8 points) Using a geometric series, the first two nonzero terms of a power series for $\frac{x}{9-x^2}$ are

A. $\frac{x}{9} + \frac{x^3}{9}$

B. $\frac{x}{9} - \frac{x^2}{27}$

C. $\frac{x}{9} + \frac{x^2}{27}$

D. $\frac{x}{9} - \frac{x^3}{81}$

E. $\boxed{\frac{x}{9} + \frac{x^3}{81}}$

Use that $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n.$

$$\frac{1}{9-x^2} = \frac{1}{9}\,\frac{1}{1-\frac{x^2}{9}} = \frac{1}{9}\sum_{n=0}^{\infty} \left(\frac{x^2}{9}\right)^n.$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{9^{n+1}}.$$

$$\frac{x}{9-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{9^{n+1}} = \frac{x}{9} + \frac{x^3}{81}.$$

7. (8 points) Find the interval of convergence for $\sum_{n=2}^{\infty}(-4)^n \frac{x^n}{2n\sqrt{\ln n}}$. $= \sum_{n=2}^{\infty} \frac{(-4x)^n}{2n\sqrt{\ln n}}$

A. $(-\frac{1}{2}, \frac{1}{2}]$

B. $[-\frac{1}{2}, \frac{1}{2})$

C. $\boxed{(-\frac{1}{4}, \frac{1}{4}]}$

D. $[-\frac{1}{4}, \frac{1}{4})$

E. $(-4, 4]$

$a_n = (-4)^n \frac{x^n}{2n\sqrt{\ln n}}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}|x|^{n+1}}{2(n+1)\sqrt{\ln(n+1)}} \cdot \frac{2n\sqrt{\ln n}}{4^n |x|^n}$

$= 4|x| \left(\frac{n}{n+1}\right)\sqrt{\frac{\ln n}{\ln n+1}} \to 4|x|$ as $n \to \infty$

$4|x| < 1$; $\boxed{|x| < 1/4}$

When $4x = 1$. $\sum_{n=2}^{\infty} \frac{(-1)^n}{2n\sqrt{\ln n}}$ converges. $4x = -1$: $\sum_{n=2}^{\infty} \frac{1}{2n\sqrt{\ln n}}$ diverges

$(-1/4, 1/4]$

8. (8 points) Find the Taylor series of $f(x) = \frac{1}{x^2+4x+6}$ centered at -2 and its radius of convergence. Notice that $x^2 + 4x + 6 = (x+2)^2 + 2$.

A. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x+2)^n$, with radius $R = 2$

B. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x+2)^n$, with radius $R = \sqrt{2}$

C. $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x+2)^{2n}$, with radius $R = \sqrt{2}$

D. $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x+2)^{2n}, \text{ with radius } R = \sqrt{2}}$

E. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x+2)^{2n}$, with radius $R = 2$

$\frac{1}{x^2+4x+6} = \frac{1}{2+(x+2)^2}$

$= \frac{1}{2} \frac{1}{1+\frac{(x+2)^2}{2}}$

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{(x+2)^2}{2}\right)^n$

$= \sum_{n=0}^{\infty}(-1)^n \frac{(x+2)^{2n}}{2^{n+1}}$

provided $\frac{(x+2)^2}{2} < 1$.

$\boxed{|x+2| < \sqrt{2}}$

Radius of conv. $\sqrt{2}$.

9. (8 points) Use the root test to find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{(1+\frac{1}{n})^{n^2}}$.

A. e

B. 1

C. $\frac{1}{e}$

D. e^2

E. $\frac{1}{e^2}$

$$\left| \frac{x^n}{(1+1/n)^{n^2}} \right|^{1/n} = \frac{|x|}{(1+1/n)^n} \longrightarrow \frac{x}{e}$$

as $n \to \infty$.

$\frac{|x|}{e} < 1$: $|x| < e$

Radius of convergence e.

10. (8 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-3)^n$ and let $F(x) = \int_3^x f(t)\,dt$. Which of the following gives an approximation of the value of $F(3.1)$ with an error less than or equal to 10^{-6}?

A. $F(3.1) \sim \frac{1}{2(10)^2} - \frac{1}{6(10)^3} + \frac{1}{12(10)^4}$

B. $F(3.1) \sim \frac{1}{2(10)^2} - \frac{1}{6(10)^3} + \frac{1}{14(10)^4}$

C. $F(3.1) \sim \frac{1}{2(10)^2} - \frac{1}{8(10)^3} + \frac{1}{15(10)^4}$

D. $F(3.1) \sim \frac{1}{2(10)^2} - \frac{1}{8(10)^3} + \frac{1}{16(10)^4}$

E. $F(3.1) \sim \frac{1}{2(10)^2} - \frac{1}{4(10)^3} + \frac{1}{6(10)^4}$

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-3)^{n+1}}{n(n+1)}.$$

$$F(3.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} \left(\frac{1}{10}\right)^{n+1}.$$

~~We~~ ~~[illegible]~~ Here

$$b_n = \frac{1}{n(n+1)} \left(\frac{1}{10}\right)^{n+1}$$

We want $b_{N+1} \leq 10^{-6}$.

or $N(N+1)\,10^{N+2} \geq 10^6$.

$\boxed{N = 3}$ We need the first 3 terms of $F(3.1)$

$$F(3.1) = \frac{1}{2(10)^2} - \frac{1}{6(10)^3} + \frac{1}{12(10)^4}.$$

11. (8 points) The first three terms of the Maclaurin expansion of $f(x) = \dfrac{\sin(x^2) - x^2}{x^6}$ are:

A. $-\frac{1}{3!} + \frac{1}{7!}x^4 - \frac{1}{9!}x^8$

B. $-\frac{1}{3!} + \frac{1}{9!}x^4 - \frac{1}{6!}x^8$

C. $-\frac{1}{3!} + \frac{1}{7!}x^4 - \frac{1}{5!}x^8$

D. $-\frac{1}{3!} + \frac{1}{5!}x^4 - \frac{1}{9!}x^8$

E. $\boxed{-\frac{1}{3!} + \frac{1}{5!}x^4 - \frac{1}{7!}x^8}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$\frac{\sin x^2 - x^2}{x^6} = -\frac{1}{3!} + \frac{x^4}{5!} - \frac{x^8}{7!} + \cdots$$

12. (4 points) If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ also always converges absolutely.

$\boxed{\text{A. True}}$
B. False

True because $\left|\frac{a_n}{n}\right| \le |a_n|$.

13. (4 points) If $a_n \geq 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ also always converges.

~~A. True~~
$\boxed{\text{B. False}}$

No. If $a_n = \frac{1}{n^2}$; $\sqrt{a_n} = \frac{1}{n}$. $\sum_{n=1}^{\infty} a_n$ converges

$\sum \sqrt{a_n}$ diverges

14. (4 points) The series $\sum_{n=1}^{\infty} (-1)^n e^{\frac{1}{n}}$ converges conditionally.

~~A. True~~
$\boxed{\text{B. False}}$

False, the series diverges.

$\lim_{n\to\infty} (-1)^n e^{1/n}$ does not exist

MA162 — EXAM III — SPRING 2017 — APRIL 11, 2017
TEST NUMBER 02

INSTRUCTIONS:

1. **Do not open the exam booklet until you are instructed to do so.**
2. **Before you open the booklet fill in the information below and use a # 2 pencil to fill in the required information on the scantron.**
3. **MARK YOUR TEST NUMBER ON YOUR SCANTRON**
4. Once you are allowed to open the exam, make sure you have a complete test. There are 8 different test pages (including this cover page).
5. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers on this test booklet.
6. There are 14 problems and the number of points each problem is worth is indicated next to the problem number. The maximum possible score is 100 points. No partial credit.
7. Do not leave the exam room during the first 20 minutes of the exam.
8. If you do not finish your exam in the first 50 minutes, you must wait until the end of the exam period to leave the room.
9. After you have finished the exam, hand in your scantron and your test booklet to your recitation instructor.

DON'T BE A CHEATER:

1. Do not give, seek or obtain any kind of help from anyone to answer questions on this exam. If you have doubts, consult only your instructor.
2. Do not look at the exam or scantron of another student.
3. Do not allow other students to look at your exam or your scantron.
4. You may not compare answers with anyone else or consult another student until after you have finished your exam, given it to your instructor and left the room.
5. Do not consult notes or books.
6. **Do not handle** phones or cameras, calculators or any electronic device until after you have finished your exam, given it to your instructor and left the room.
7. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs collect the scantrons and the exams.
8. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above statements regarding academic dishonesty:

STUDENT NAME: SOLUTIONS

STUDENT SIGNATURE: ______________________

STUDENT ID NUMBER: ______________________

SECTION NUMBER AND RECITATION INSTRUCTOR: ______________________

FORMULA SHEET

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{for all } x.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x.$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \text{ for all } x.$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \frac{k(k-1)(k-2)\dots(k-n+1)}{n!} x^n \dots$$
$$\text{for } |x| < 1.$$

If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$, for $|x-a| < R$, then $\int_a^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-a)^{n+1}$ for $|x-a| < R$.

If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$, for $|x-a| < R$, then $f'(x)\, dt = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$ for $|x-a| < R$.

1. (8 points) Consider the two series

$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{5^n} \text{ and } (ii)\ \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}.$$

$$(i)\ \sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1/5}{1-1/5}$$

Converges

Which of the following is true?

A. (i) and (ii) converge conditionally

(i) converges absolutely

B. (i) converges conditionally and (ii) converges absolutely

C. (i) converges absolutely and (ii) converges conditionally

(ii) Converges because it's an alternating series.

D. (i) and (ii) converge absolutely

E. (i) and (ii) diverge

However.

$$\frac{n}{\sqrt{n^3+1}} \sim \frac{n}{n^{3/2}} = 1/n^{1/2}$$

when n is large so the series diverges by the limit comparison thm.

2. (8 points) The series $\sum_{n=1}^{\infty} \frac{1}{n(\ln(2n))}$

A. Converges by the ratio test

B. Diverges by the ratio test

C. Converges by the root test

D. Converges by the integral test

E. Diverges by the integral test

$$\int_1^{\infty} \frac{1}{x(\ln 2x)}\, dx \qquad \ln 2x = u \qquad du = \frac{d}{x}$$

$$= \int_{\ln 2}^{\infty} \frac{1}{u}\, du \text{ diverges.}$$

the root and ratio tests are inconclusive.

3. (8 points) Which of the following series are convergent?

I. $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$

II. $\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^2}$

III. $\sum_{n=1}^{\infty} \frac{3n^2}{(n^3+1)^{2/3}}$

A. $\boxed{\text{II only}}$

B. I and II only

C. II and III only

D. None of them

E. All of them

$\frac{3n^2}{n^3+1} \sim \frac{3}{n}$ when n is large

Series diverges by the limit Comparison theorem

$\frac{3n^2}{(n^3+1)^2} \sim \frac{3n^2}{n^6} = 3/n^4$. for large n.

Converges

$\frac{3n^2}{(n^3+1)^{2/3}} \sim \frac{3n^2}{n^2} = 3$ for n large.

Series diverges.

4. (8 points) For what values of a is the series $\sum_{n=1}^{\infty} \left(\frac{an}{n+1}\right)^n$ absolutely convergent?

A. $|a| \leq 1$

B. $|a| = 1$

C. $|a| \geq 1$

D. $|a| > 1$

E. $\boxed{|a| < 1}$

$\lim_{n \to \infty} \left| \frac{an}{n+1} \right| = |a|$, the Series converges absolutely if $|a| < 1$ by the root test.

When $\underline{a = 1}$ $\left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e}$ as $n \to \infty$

~~Case 1~~ the Series diverges.

when $\underline{a = -1}$ $\left(\frac{-n}{n+1}\right)^n = (-1)^n \frac{1}{(1+1/n)}$ does not have a limit.

5. Compute the Taylor series of $f(x) = \ln x$ centered at 5 and use it to find $f^{(10)}(5)$.

A. $f^{(10)}(5) = \frac{7!}{5^{10}}$.

B. $f^{(10)}(5) = -\frac{10!}{5^{10}}$.

C. $f^{(10)}(5) = \frac{10!}{5^{10}}$.

D. $f^{(10)}(5) = \frac{9!}{5^{10}}$.

E. $\boxed{f^{(10)}(5) = -\frac{9!}{5^{10}}}$.

$$\frac{1}{x} = \frac{1}{5+x-5} = \frac{1}{5}\frac{1}{1+\frac{x-5}{5}} = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^n} = \sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}}.$$

$$\ln x = C + \int\frac{dx}{x} = C + \sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^{n+1}}{(n+1)5^{n+1}}$$

Set $x = 5$, $C = \ln 5$.

$$\ln x = \ln 5 + \sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^{n+1}}{(n+1)5^{n+1}}.$$ The coefficient of $(x-5)^{10}$ is $\frac{f^{(10)}(5)}{10!}$ so $\frac{f^{(10)}(5)}{10!} = (-1)^9\frac{1}{10 \cdot 5^{10}}$ $f^{(10)}(5) = -\frac{9!}{5^{10}}$

6. (8 points) Using a geometric series, the first two nonzero terms of a power series for $\frac{2x}{4+x^2}$ are

A. $\boxed{\frac{x}{2} - \frac{x^3}{8}}$

B. $\frac{x}{2} + \frac{x^3}{8}$

C. $\frac{x}{2} + \frac{x^3}{2}$

D. $\frac{x}{2} - \frac{x^2}{32}$

E. $\frac{x}{2} + \frac{x^2}{32}$

$$\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\frac{x^2}{4}} = \frac{1}{4}\left(1 - \frac{x^2}{4} + \frac{x^4}{16} - \dots\right)$$

$$= \frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64} - \dots$$

$$\frac{2x}{4+x^2} = \frac{2x}{4} - \frac{2x^3}{16} + \frac{2x^5}{64} + \dots$$

$$= \frac{x}{2} - \frac{x^3}{8} + \dots$$

7. (8 points) Find the interval of convergence for $\sum_{n=2}^{\infty} \frac{3 \cdot 9^n}{n\sqrt{\ln n}} x^n$. $= \sum_{n=2}^{\infty} \frac{3\,(9x)^n}{n\sqrt{\ln n}}$.

A. $(-\frac{1}{3}, \frac{1}{3}]$

B. $[-\frac{1}{3}, \frac{1}{3})$

C. $(-\frac{1}{9}, \frac{1}{9}]$

D. $\boxed{[-\frac{1}{9}, \frac{1}{9})}$

E. $(-9, 9]$

$$a_n = \frac{3\,(9x)^n}{n\sqrt{\ln n}} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{3\,(9x)^{n+1}}{(n+1)\sqrt{\ln(n+1)}} \frac{n\sqrt{\ln n}}{3(9x)^n}$$

$$= |9x| \frac{n}{n+1} \sqrt{\frac{\ln n}{\ln n+1}} \to |9x| \text{ as } n \to \infty$$

We need $|9x| < 1$ $|x| < 1/9$.

End points: $9x = 1$; $\sum_{n=2}^{\infty} \frac{3}{n \ln n}$ diverse

$9x = -1$; $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ conv $[-1/9, 1/9)$

8. (8 points) Find the Taylor series of $f(x) = \frac{1}{x^2 + 4x + 7}$ centered at -2 and its radius of convergence. Notice that $x^2 + 4x + 7 = (x+2)^2 + 3$.

A. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(x+2)^n$, with radius $R = 3$

B. $f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n}(x+2)^n$, with radius $R = \sqrt{3}$

C. $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x+2)^{2n}, \text{ with radius } R = \sqrt{3}}$

D. $f(x) = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+2)^{2n}$, with radius $R = \sqrt{3}$

E. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x+2)^{2n}$, with radius $R = 3$

$$\frac{1}{x^2+4x+7} = \frac{1}{3+(x+2)^2}$$

$$= \frac{1}{3} \frac{1}{1+\frac{(x+2)^2}{3}} =$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left[\frac{(x+2)^2}{3}\right]^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x+2)^{2n}}{3^{n+1}}$$

provided $\frac{(x+2)^2}{3} < 1$ $|x+2| < \sqrt{3}$

Radius of convergence: $\sqrt{3}$

9. (8 points) Use the root test to find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(1+\frac{1}{n})^{n^2}}{2^n} x^n$.

A. $2e$

B. 2

C. $\frac{1}{2e}$

D. $\frac{e^2}{2}$

E. $\boxed{\frac{2}{e}}$

$$a_n = \frac{(1+1/n)^{n^2}}{2^n} x^n.$$

$$|a_n|^{1/n} = \frac{|x|}{2} \cdot (1+1/n)^n \to e \frac{|x|}{2} < 1$$

$$|x| < 2/e.$$

Radius of convergence $2/e$.

10. (8 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n}(x-3)^n$ and let $F(x) = \int_3^x f(t)\,dt$. Which of the following gives an approximation of the value of $F(3.1)$ with an error less than or equal to 10^{-6}?

A. $F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{12(10)^3} + \frac{1}{36(10)^4}$

B. $\boxed{F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{12(10)^3} + \frac{1}{24(10)^4}}$

C. $F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{8(10)^3} + \frac{1}{15(10)^4}$

D. $F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{8(10)^3} + \frac{1}{16(10)^4}$

E. $F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{4(10)^3} + \frac{1}{6(10)^4}$

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(n+1)} (x-3)^{n+1}.$$

$$F(3.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(n+1)} \left(\frac{1}{10}\right)^{n+1}$$

$$b_n = \frac{1}{2n(n+1)} \left(\frac{1}{10}\right)^{n+1}$$

We need $b_{n+1} \leq 10^{-6}$

$$2(n+1)(n+2)10^{n+2} \geq 10^6 ; \quad n=3$$

We need the first 3 terms

$$F(3.1) \sim \frac{1}{4(10)^2} - \frac{1}{12(10)^3} + \frac{1}{24(10)^4}.$$

11. (8 points) The first three terms of the Maclaurin expansion of $f(x) = \dfrac{\cos(x^2) - 1}{x^4}$ are:

A. $-\frac{1}{2!} + \frac{1}{4!}x^4 - \frac{1}{4!}x^8$

B. $-\frac{1}{2!} + \frac{1}{4!}x^4 + \frac{1}{8!}x^8$

C. $\boxed{-\frac{1}{2!} + \frac{1}{4!}x^4 - \frac{1}{6!}x^8}$

D. $-\frac{1}{2!} + \frac{1}{6!}x^4 - \frac{1}{10!}x^8$

E. $-\frac{1}{2!} + \frac{1}{6!}x^4 - \frac{1}{8!}x^8$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$$

$$\cos x^2 - 1 = -\frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} \cdots$$

$$\frac{\cos x^2 - 1}{x^4} = -\frac{1}{2!} + \frac{x^4}{4!} - \frac{x^8}{6!} + \cdots$$

12. (4 points) If a series $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ also always diverges.

A. True

B. $\boxed{\text{False}}$

No. If $a_n = 1/n$; $\frac{a_n}{n} = 1/n^2$. $\sum a_n$ diverges, $\sum \frac{a_n}{n}$ converges

13. (4 points) If $a_n \geq 0$ and the series $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n^2$ also always diverges.

A. True

B. $\boxed{\text{False}}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. False

14. (4 points) The series $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{1}{n})$ diverges.

A. $\boxed{\text{True}}$

B. False

Yes, $\lim_{n \to \infty} (-1)^n \cos(1/n)$ does not exist

So the series diverges.