Answer Keys for Exam I Version 01

1. $g(t) = \sqrt{3-t} + \dfrac{1}{\sqrt{6+t}}$

Domain of $\sqrt{3-t}$: $3 - t \geq 0$

i.e. $3 \geq t$

i.e. $(-\infty, 3]$

Domain of $\dfrac{1}{\sqrt{6+t}}$: $6 + t \geq 0$ & $6 + t \neq 0$

i.e. $6 + t > 0$

i.e. $t > -6$.

i.e. $(-6, \infty)$

Therefore, we conclude

Domain of $g(t)$:

$= (\text{Domain of } \sqrt{3-t}) \cap (\text{Domain of } \dfrac{1}{\sqrt{6+t}})$

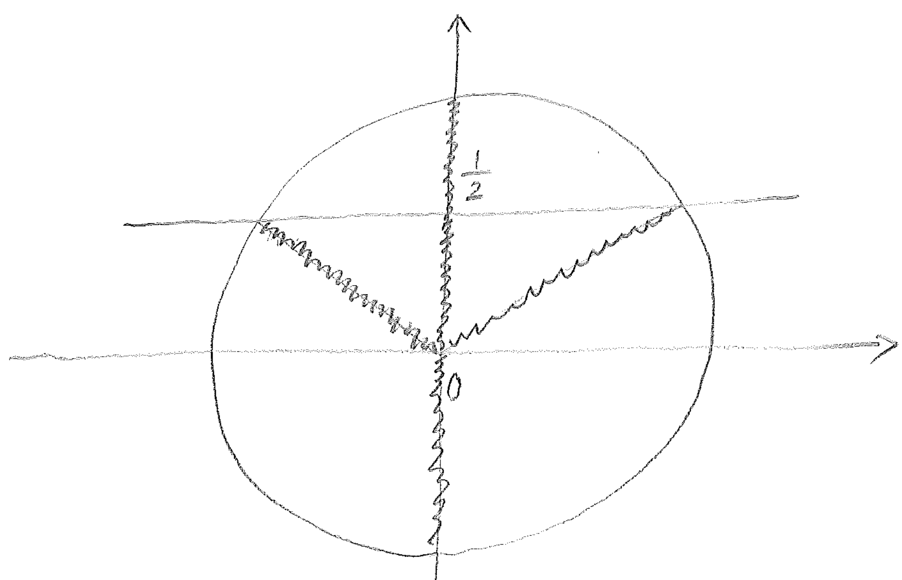
$= (-\infty, 3] \cap (-6, \infty) = (-6, 3]$

$\boxed{\text{Answer D. } (-6, 3]}$

2.
$$\sin(2x) = \cos x$$
$$\parallel$$
$$2 \sin x \, co x$$

$$\therefore \quad 2 \sin x \, co x - \cos x$$
$$= \cos x \, (2 \sin x - 1) = 0$$
$$\therefore \quad \cos x = 0 \quad \text{or} \quad 2 \sin x - 1 = 0$$

i.e.
$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$



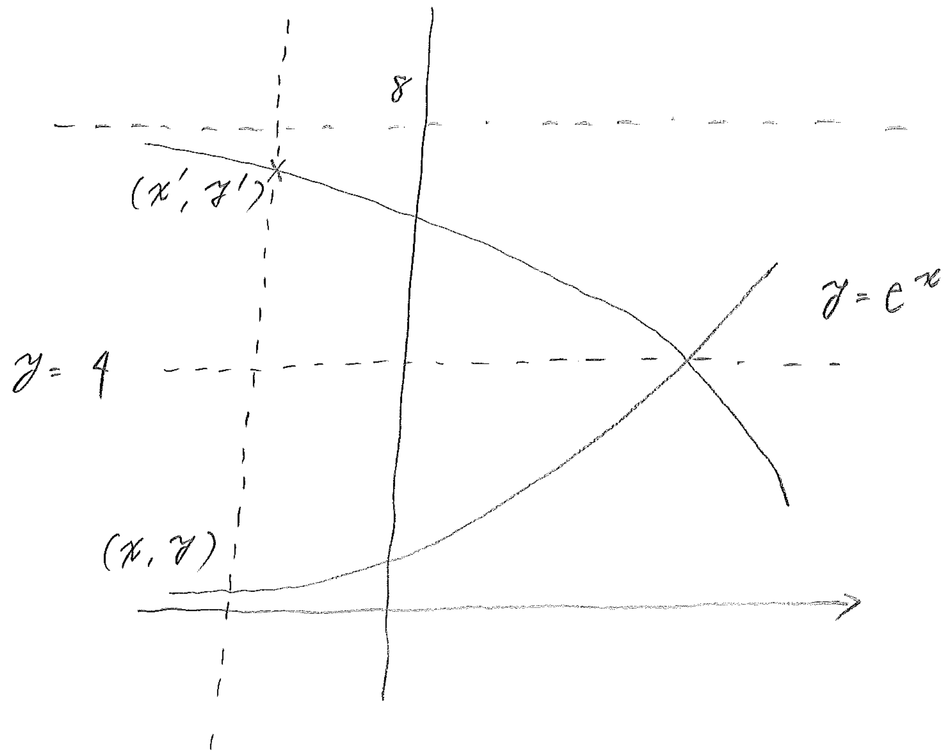
The values of x in the interval $[0, 2\pi]$ satisfying

$\cos x = 0$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = \frac{1}{2}$ $\quad x = \frac{\pi}{6}, \frac{5\pi}{6}$.

$\boxed{\text{Answer E. 4.}}$

3.



Let (x', y') be the point obtained by reflecting the point (x, y) ~~on~~ the graph $y = e^x$ about the line $y = 4$

Then $\begin{cases} x' = x \\ \dfrac{y' + y}{2} = 4 \quad \text{i.e. } y' = 8 - y \end{cases}$

Therefore, we conclude (x', y') is on the graph of

$$y' = 8 - y = 8 - e^x = 8 - e^{x'}$$

i.e.,

$$y' = 8 - e^{x'}$$

Replacing (x', y') with (x, y).

we obtain

$$y = 8 - e^x$$

Answer A. $y = 8 - e^x$

4.

$$\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \frac{\cos x}{\sin x}$$

We observe that

$$\lim_{x \to \pi^-} \cos x = -1.$$

and that

when x approaches π from the left

(i.e. $x \to \pi^-$)

$\sin x$ appoaches 0 from the right.

(i.e. $\sin x \to 0^+$)

Therefore

$$\lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

$\boxed{\text{Answer B. } -\infty.}$

5. $\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$

$$= \lim_{h \to 0} \frac{(\sqrt{9+h} - 3)(\sqrt{9+h} + 3)}{h(\sqrt{9+h} + 3)}$$

$$= \lim_{h \to 0} \frac{\overbrace{(\sqrt{9+h})^2 - 3^2}^{= \cancel{h}}}{\cancel{h}(\sqrt{9+h} + 3)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{6}$$

$$\boxed{\text{Answer C. } \frac{1}{6}}$$

6.

$$\lim_{x \to \infty} \left(e^{-x} + 2\cos(3x) \right)$$

Observe that

$$\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^{x}} = 0$$

and that

$$\lim_{x \to \infty} 2\cos(3x) \text{ does not exist},$$

since $2\cos(3x)$ oscillates as x approaches ∞.

Therefore,

$$\lim_{x \to \infty} \left(e^{-x} + 2\cos(3x) \right)$$

does not exist.

$$\boxed{\text{Answer E. Does Not Exist.}}$$

7.

In order for f to be continuous, we have to have

$$\lim_{x \to a} f(x) = f(a).$$

In particular, we have

$$\begin{array}{ccc} \lim_{x \to a^+} f(x) & = & f(a). \\ \| & & \| \\ \lim_{x \to a^+} (3x+1) & & a+2. \\ \| & & \\ 3a+1. & & \end{array}$$

That is to say, we have to have.

$$3a + 1 = a + 2.$$

$$\therefore \quad a = \frac{1}{2}.$$

On the other hand, it is easy to check for $a = \frac{1}{2}$ the function f is continuous.

$$\boxed{\text{Answer} \quad B. \quad \frac{1}{2}.}$$

8.

(i) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-2 + x) = 0$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0$

Therefore

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 0$,

and hence

$\lim_{x \to 2} f(x) = 0$ exists.

Note. $\lim_{x \to 2} f(x) = 0 \neq f(2) = 2$.

(ii) Since

$\lim_{x \to 2} f(x) = 0 \neq f(2)$,

f is NOT continuous at $x = 2$.

(iii) Since f is NOT continuous at $x = 2$, f is NOT differentiable at $x = 2$.

$\boxed{\text{Answer A. (i) only}}$

9. The tangent to the curve $y = x\sqrt{x} = x^{\frac{3}{2}}$ is parallel to the line $y = 1 + x$. with slope 1.

Therefore,

$$y' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2} = 1.$$

Solving for x, we have.

$$x = \frac{4}{9}.$$

$$y = x\sqrt{x} = \frac{8}{27}$$

Therefore, the equation of the tangent line is.

$$y - \frac{8}{27} = 1 \cdot \left(x - \frac{4}{9}\right)$$

i.e.

$$y = x - \frac{4}{27}.$$

$$\boxed{\text{Answer D. } \quad y = x - \frac{4}{27}.}$$

10.

$$f(x) = e^x g(x)$$

$$f'(x) = e^x g(x) + e^x g'(x).$$

Therefore, we have

$$f'(0) = e^0 g(0) + e^0 g'(0)$$

$$= g(0) + g'(0)$$

$$= 3 + 10 = 13.$$

$$\boxed{\text{Answer C. } f'(0) = 13}$$

11.

$$f(x) = \frac{1-x}{1+x}$$

$$f'(x) = \frac{(-1)(1+x) - (1-x)\cdot 1}{(1+x)^2}$$

$$= \frac{-2}{(1+x)^2} = -2(1+x)^{-2}$$

$$f''(x) = (-2)(-2)(1+x)^{-3}$$

$$= \frac{4}{(1+x)^3}$$

$$\boxed{\text{Answer} \quad A.\ \frac{4}{(1+x)^3}}$$

12.

We have

$$-1 \leq \sin\left(\frac{2x}{\pi}\right) \leq 1.$$

Multiply $\frac{1}{x}$ (since $x \to \infty$, we may assume $\frac{1}{x} > 0$).

to have

$$-\frac{1}{x} \leq \frac{1}{x}\sin\left(\frac{2x}{\pi}\right) \leq \frac{1}{x}.$$

$x \to \infty$: $-\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$.

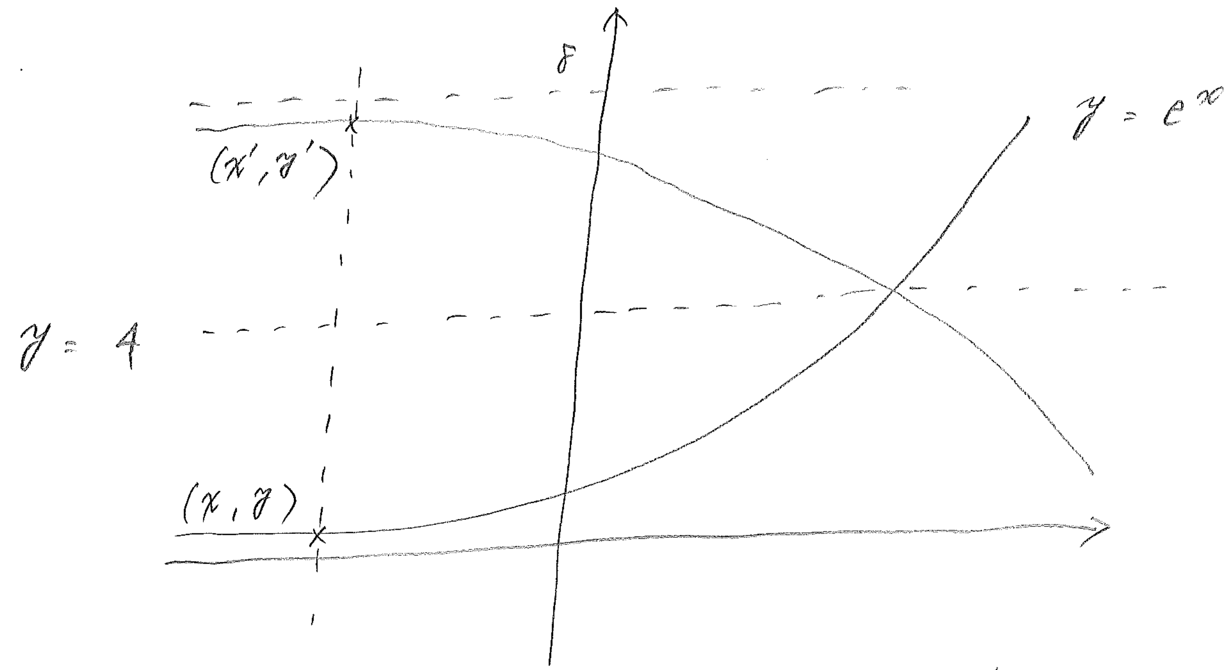
$\therefore$ $\frac{1}{x}\sin\left(\frac{2x}{\pi}\right) \to 0$ by Squeeze Theorem.

$$\therefore \lim_{x \to \infty} \frac{1}{x}\sin\left(\frac{2x}{\pi}\right) = 0$$

$\boxed{\text{Answer D. } 0}$

Answer Keys for Exam 1 Version 02

1.



Let (x', y') be the point obtained by reflecting the point (x, y) on the graph $y = e^x$ about the line $y = 4$.

Then
$$\begin{cases} x' = x \\ \dfrac{y'+y}{2} = 4 \quad \text{i.e.,} \quad y' = 8 - y \end{cases}$$

Therefore, we conclude (x', y') is on the graph of

$$y' = 8 - y = 8 - e^x = 8 - e^{x'}$$

i.e. $y' = 8 - e^{x'}$.

Replacing (x', y') with (x, y),

we obtain

$$y = 8 - e^x$$

Answer F, $y = 8 - e^x$

2. $g(t) = \sqrt{5-t} + \dfrac{1}{\sqrt{7+t}}$

Domain of $\sqrt{5-t}$: $5 - t \geq 0$

i.e. $5 \geq t$

i.e. $(-\infty, 5]$

Domain of $\dfrac{1}{\sqrt{7+t}}$: $7 + t \geq 0$

& $7 + t \neq 0$

i.e. $7 + t > 0$

i.e. $t > -7$

i.e. $(-7, \infty)$

Therefore, we conclude

Domain of $g(t)$:

$(\text{Domain of } \sqrt{5-t}) \cap (\text{Domain of } \dfrac{1}{\sqrt{7+t}})$

$= (-\infty, 5] \cap (-7, \infty) = (-7, 5]$

$\boxed{\text{Answer B}, (-7, 5]}$

3.

$$\underset{\underset{2\sin x\cos x}{\shortparallel}}{\sin(2x)} = \cos x$$

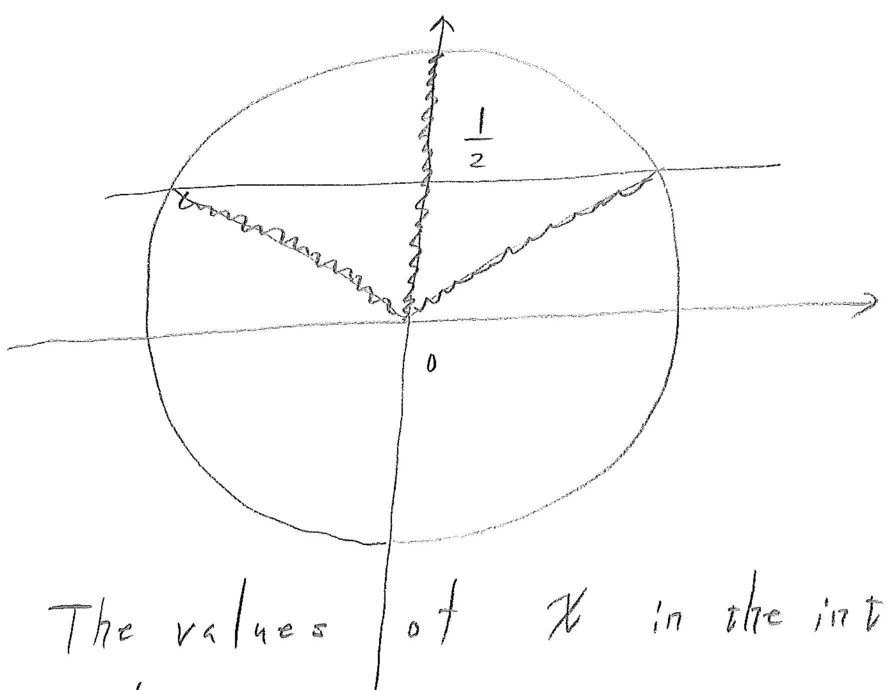
$\therefore$ $2\sin x\cos x - \cos x$

$= \cos x\,(2\sin x - 1) = 0$

$\therefore$ $\cos x = 0$ or $2\sin x - 1 = 0$

i.e.

$\cos x = 0$ or $\sin x = \frac{1}{2}$



The values of x in the interval $[0, \pi]$ satisfying

$\cos x = 0$ $\quad$ $x = \frac{\pi}{2}$

$\sin x = \frac{1}{2}$ $\quad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\boxed{\text{Answer D. } 3}$

4. $$\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \frac{\cos x}{\sin x}$$

We observe that

$$\lim_{x \to \pi^-} \cos x = -1$$

and that

when x approaches π from the left

(i.e. $x \to \pi^{-1}$)

$\sin x$ approaches 0 from the right

(i.e. $\sin x \to 0^+$)

Therefore,

$$\lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

$\boxed{\text{Answer B. } -\infty}$

5.

$$\lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{16+h} - 4)(\sqrt{16+h} + 4)}{h(\sqrt{16+h} + 4)}$$

$$= \lim_{h \to 0} \frac{\overbrace{(\sqrt{16+h})^2 - 4^2}^{= \cancel{h}}}{\cancel{h}(\sqrt{16+h} + 4)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{16+h} + 4} = \frac{1}{8}$$

$$\boxed{\text{Answer C. } \frac{1}{8}}$$

6. $$\lim_{x \to \infty} \left(e^{-x} + 2\cos(3x) \right)$$

Observe that

$$\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^{x}} = 0$$

and that

$\lim_{x \to \infty} 2\cos(3x)$ does not exist,

since $2\cos(3x)$ oscillates as x approaches ∞

Therefore,

$$\lim_{x \to \infty} \left(e^{-x} + 2\cos(3x) \right)$$

does not exist.

Answer. E. Does Not Exist.

7. In order for f to be continuous, we have to have

$$\lim_{x \to a} f(x) = f(a)$$

In particular, we have

$$\begin{array}{ccc} \displaystyle\lim_{x \to a^+} f(x) & = & f(a) \\ \| & & \| . \\ \displaystyle\lim_{x \to a^+} (2x+1) & & a+2. \\ \| & & \\ 2a+1. & & \end{array}$$

That is to say, we have to have

$$2a + 1 = a + 2.$$

$$\therefore \quad a = 1.$$

On the other hand, it is easy to check for $a = 1$ the function f is continuous.

$$\boxed{\text{Answer D. 1.}}$$

8.

(i) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-4 + x) = 1.$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2 - x) = 1.$

Therefore,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 1,$

and hence

$\lim_{x \to 3} f(x) = 1$ exists.

Note. $\lim_{x \to 3} f(x) = 1 \neq f(3) = 2$

(ii) Since

$\lim_{x \to 3} f(x) = 1 \neq f(3),$

f is NOT continuous at $x = 3$

(iii) Since f is NOT continuous at $x = 3$, f is NOT differentiable at $x = 3$.

$\boxed{\text{Answer D. (i) only}}$

9. The tangent to the curve $y = x\sqrt{x} = x^{\frac{3}{2}}$ is parallel to the line $y = 1 + x$ with slope 1.

Therefore,

$$y' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2} = 1$$

Solving for x, we have

$$x = \frac{4}{9}$$

$$y = x\sqrt{x} = \frac{8}{27}.$$

Therefore, the equation of the tangent line is

$$y - \frac{8}{27} = 1 \cdot \left(x - \frac{4}{9} \right)$$

i.e.

$$y = x - \frac{4}{27}$$

$$\boxed{\text{Answer E. } \quad y = x - \frac{4}{27}}$$

10.

$$f(x) = e^x g(x)$$

$$f'(x) = e^x g(x) + e^x g'(x)$$

Therefore, we have

$$f'(0) = e^0 g(0) + e^0 g'(0)$$

$$= 5 + 7 = 12.$$

$$\boxed{\text{Answer A} \quad f'(0) = 12.}$$

11.

$$f(x) = \frac{1 - x}{1 + x}$$

$$f'(x) = \frac{(-1)(1+x) - (1-x)\cdot 1}{(1+x)^2}$$

$$= \frac{-2}{(1+x)^2} = -2(1+x)^{-2}$$

$$f''(x) = (-2)(-2)(1+x)^{-3}$$

$$= \frac{4}{(1+x)^3}$$

$$\boxed{\text{Answer} \quad E. \quad \frac{4}{(1+x)^3}}$$

12. We have

$$-1 \leqq \cos\left(\frac{2x}{\pi}\right) \leqq 1.$$

Multiply $\frac{1}{x}$ $\left(\text{since } x \to \infty, \text{ we may assume } \frac{1}{x} > 0\right)$ to have

$$-\frac{1}{x} \leqq \frac{1}{x}\cos\left(\frac{2x}{\pi}\right) \leqq \frac{1}{x}$$

$x \to \infty$: $-\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$

$\therefore$ $\frac{1}{x}\cos\left(\frac{2x}{\pi}\right) \to 0$ by Squeeze Theorem.

$$\therefore \lim_{x \to \infty} \frac{1}{x}\cos\left(\frac{2x}{\pi}\right) = 0$$

$\boxed{\text{Answer A. } 0.}$