NAME GRADING KEY

STUDENT ID

RECITATION INSTRUCTOR

RECITATION TIME

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TOTAL	/100

DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4 .
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Find the integrals in problems 1-5.

(6) 1.(a) $\int \sin^3 x \cos^2 x dx. = \int (1-\cos^2 x)\cos^2 x \sin x\, dx =$

$u = \cos x \quad du = -\sin x\, dx$

$= -\int (1-u^2)u^2\, du = \int (-u^2+u^4)\, du$

$= -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

−1 pt for missing +C (1 time only for test)

−1 pt for missing dx, du, etc (1 time only for each problem)

3 3

$\boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$ $\boxed{6}$

(6) 2. $\int \tan^3 x \sec^3 x dx = \int \tan^2 x \sec^2 x (\sec x \tan x) dx$

$= \int (\sec^2 x - 1)\sec^2 x (\sec x \tan x)\, dx$

$u = \sec x \quad du = \sec x \tan x\, dx$

$= \int (u^2-1)u^2\, du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

3 3

$\boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$ $\boxed{6}$

(6) 3. $\displaystyle\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\,dx.$ $\displaystyle = \int_0^{\frac{\pi}{4}} u^2\,du = \frac{u^3}{3}\Big|_0^{\frac{\pi}{4}} = \frac{\pi^3}{3\cdot 4^3} = \frac{\pi^3}{192}$

$u = \tan^{-1}x \quad du = \frac{1}{1+x^2}\,dx$

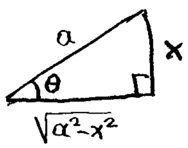
$x = 0 \rightarrow u = 0$

$x = 1 \rightarrow u = \frac{\pi}{4}$

or: $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx = \frac{1}{3}(\tan^{-1}x)^3\Big|_0^1 = \frac{1}{3}\left(\frac{\pi}{4}\right)^3 = \frac{\pi^3}{192}$

$\boxed{\dfrac{\pi^3}{192}}$ $\boxed{6}$

(15) 4. $\displaystyle\int \frac{x^2}{(a^2-x^2)^{\frac{3}{2}}}\,dx,$ where a is a positive constant.



$x = a\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = a\cos\theta\,d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

$\displaystyle = \int \frac{a^2\sin^2\theta}{a^3\cos^3\theta}\,a\cos\theta\,d\theta$

0 pts for problem if the integrand is wrong, but -2 pts for wrong power of a, and grade rest with consistency

$\displaystyle = \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$\displaystyle = \int(\sec^2\theta - 1)\,d\theta$

-3pts for $1-\sec^2\theta$ and grade rest with consistency.

$= \tan\theta - \theta + C$

$\displaystyle = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$

$\boxed{\dfrac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\dfrac{x}{a} + C}$ $\boxed{15}$

(15) 5. $\displaystyle\int \frac{3x^2-1}{(x-1)(x^2+1)}\,dx$

$\displaystyle\frac{3x^2-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

No additional credit beyond this point if anything is wrong here

$3x^2 - 1 = Ax^2 + A + Bx^2 + Cx - Bx - C$

$$\left.\begin{aligned} A + B &= 3 \\ -B + C &= 0 \\ A \quad - C &= -1 \end{aligned}\right] (+) \rightarrow 2A = 2 \quad \boxed{A=1} \ \boxed{B=2} \ \boxed{C=2}$$

$\displaystyle\int \frac{3x^2-1}{(x-1)(x^2+1)} = \int\left[\frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\right]dx$

$= \ln|x-1| + \ln|x^2+1| + 2\tan^{-1}x + C$

$\boxed{\ln|x-1| + \ln|x^2+1| + 2\tan^{-1}x + C}$ $\boxed{15}$

(6) 6. Make a substitution $u = \ldots$ to express the integral $\int \frac{1}{x\sqrt{x+1}}\,dx$ as an integral of a rational function in u. Do not evaluate the integral.

$$\int \frac{1}{x\sqrt{x+1}}\,dx = \int \frac{1}{(u^2-1)\,u}\,2u\,du$$

$u = \sqrt{x+1}$ 2

$u^2 = x+1$

$x = u^2 - 1$

$dx = 2u\,du$

4

$$\boxed{\int \frac{1}{x\sqrt{x+1}}\,dx = \int \frac{2}{u^2-1}\,du}$$ $\boxed{6}$

(12) 7. Determine whether each integral is convergent or divergent. Find its value if it is convergent. Important: You must show clearly how limits are involved.

(a) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx$ 3

$$= \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}(\ln t)^2\right] = \infty$$ 2

1

$\boxed{\text{divergent}}$ $\boxed{6}$

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$ 3

$$= \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-} \underbrace{\sin^{-1}t}_{2} = \frac{\pi}{2}$$

1

$\boxed{\frac{\pi}{2}}$ $\boxed{6}$

(12) 8. Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \frac{\pi}{4}$.

[Note: $\frac{d}{dx}\ln(\sec x + \tan x) = \sec x$].

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \overset{4}{=} \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{1}{\sec x}\sec x\tan x\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\,dx = \int_0^{\frac{\pi}{4}} \underbrace{\sec x}_{4}\,dx$$

2

$$= \ln(\sec x + \tan x)\Big|_0^{\frac{\pi}{4}}$$

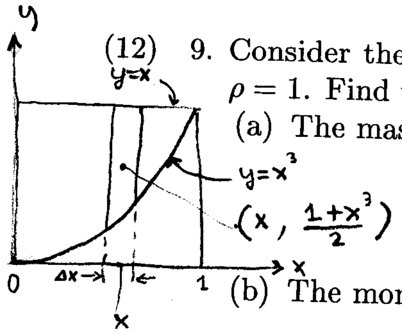
$$= \ln\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)$$

$$= \ln(\sqrt{2}+1) - \ln 1$$

$$= \ln(\sqrt{2}+1)$$

2

$\boxed{\ln(\sqrt{2}+1)}$ $\boxed{12}$



(12) 9. Consider the lamina bounded by the curves $y = x^3$, $y = 1$, $x = 0$ and with density $\rho = 1$. Find the following:

For (a), (b), (c) 0 points if answer is correct but no work is shown

(a) The mass m of the lamina.

$$m = \int_0^1 1\left(1 - x^3\right)dx = \left(x - \frac{x^4}{4}\right)\Bigg|_0^1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$\boxed{m = \frac{3}{4}}$ 2 NPC

(b) The moment M_y of the lamina about the y-axis.

$$M_y = \int_0^1 x \cdot 1 \cdot (1 - x^3)\,dx = \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

$\boxed{M_y = \frac{3}{10}}$ 3 NPC

(c) The moment M_x of the lamina about the x-axis.

$$M_x = \int_0^1 \frac{1 + x^3}{2} \cdot 1 \cdot \left(1 - x^3\right) dx = \frac{1}{2}\int_0^1 (1 - x^6)\,dx = \frac{1}{2}\left(x - \frac{x^7}{7}\right)\Bigg|_0^1 = \frac{1}{2}\left(1 - \frac{1}{7}\right) = \frac{3}{7}$$

$\boxed{M_x = \frac{3}{7}}$ 3 NPC

(d) The center of mass $(\overline{x}, \overline{y})$ of the lamina.

$$m\overline{x} = M_y \rightarrow \frac{3}{4}\overline{x} = \frac{3}{10} \rightarrow \overline{x} = \frac{2}{5}$$

$$m\overline{y} = M_x \rightarrow \frac{3}{4}\overline{y} = \frac{3}{7} \rightarrow \overline{y} = \frac{4}{7}$$

2 2

$\boxed{(\overline{x}, \overline{y}) = \left(\frac{2}{5}, \frac{4}{7}\right)}$ ok if consistent with above answers

(10) 10. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem).

2 pts each NPC

(a) $a_n = \dfrac{3^n}{4^{n+2}}$

$$\lim_{n \to \infty} a_n = \frac{1}{16} \lim_{n \to \infty} \left(\frac{3}{4}\right)^n = 0$$

$\left|\frac{3}{4}\right| < 1$

$\boxed{0}$ 2

(b) $\left\{\dfrac{3 + (-1)^n}{n^2}\right\}$

squeeze theorem

$$\frac{2}{n^2} \le \frac{3 + (-1)^n}{n^2} \le \frac{4}{n^2}$$

$\downarrow$ $\therefore\downarrow$ $\downarrow$ as $n \to \infty$

0 0 0

$\boxed{0}$ 2

(c) $a_n = \dfrac{\sqrt{n}}{\ln n}$

$$\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \underset{\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\sqrt{x}}{2} = \infty$$

$\therefore$ sequence diverges

$\boxed{\text{diverges}}$ 2

(d) $a_n = \dfrac{n^3}{1 + n^2}$

$$\lim_{n \to \infty} \frac{n^3}{1 + n^2} = \lim_{n \to \infty} \frac{n}{\frac{1}{n^2} + 1} = \infty$$

$\boxed{\text{diverges}}$ 2

(e) $a_n = \dfrac{(-1)^n n!}{(n+1)!}$

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0$$

$\therefore \lim_{n \to \infty} a_n = 0$

$\boxed{0}$ 2