NAME GRADING KEY

10–DIGIT PUID ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

Page 1	/18
Page 2	/28
Page 3	/32
Page 4	/22
TOTAL	/100

DIRECTIONS

1. Write your name, 10–digit PUID, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

Find the integrals in problems 1–5.

−1 pt for missing +C (1 time for test)
−1 pt for missing dx, du etc (1 time for each problem)

(8) 1. $\int_0^{\frac{\pi}{4}} \sec^4 x\,dx = \int_0^{\frac{\pi}{4}} \sec^2 x\,(\sec^2 x\,dx)$

4 $= \int_0^1 (1+u^2)\,du$ (letting $u = \tan x$, $du = \sec^2 x\,dx$, $u(0)=0$, $u(\frac{\pi}{4})=1$)

$= \left[u + \frac{u^3}{3}\right]_0^1$

$= \frac{4}{3}$

4

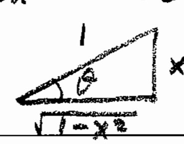
$\boxed{\frac{4}{3}}$

$\boxed{8}$

(10) 2. $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx = \int \frac{x^2}{(\sqrt{1-x^2})^3}\,dx$

4 $= \int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta\,d\theta$ ($x = \sin\theta$, $dx = \cos\theta\,d\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$)

$\sqrt{1-x^2} = \cos\theta$



$= \int \frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta$

$= \int (\sec^2\theta - 1)\,d\theta$

$= \tan\theta - \theta + C$

$= \frac{x}{\sqrt{1-x^2}} - \sin^{-1}x + C$

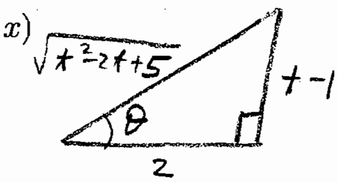
$\boxed{\frac{x}{\sqrt{1-x^2}} - \sin^{-1}x + C}$

3 3

$\boxed{10}$

(−3 pt if answer is left in terms of θ)

(10) 3. $\int \frac{dt}{\sqrt{t^2 - 2t + 5}}$ (Hint: $\frac{d}{dx} \ln|\sec x + \tan x| = \sec x$)



3 $= \int \frac{dt}{\sqrt{(t-1)^2 + 2^2}}$

$= \int \frac{2\sec^2\theta}{\sqrt{4\tan^2\theta + 4}}\, d\theta$ $\left(t-1 = 2\tan\theta,\ dt = 2\sec^2\theta\, d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right)$

3 $= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left| \frac{\sqrt{t^2-2t+5}}{2} + \frac{t-1}{2} \right| + C$ ← or →

$\boxed{\ln\left|\sqrt{t^2-2t+5} + (t-1)\right| + C}$

2 2

(−2 pts if answer is left in terms of θ) $\boxed{10}$

(10) 4. $\int \frac{dx}{x(1 + \sqrt{x})}$

(Hint: First make a substitution to express the integrand as a rational function)

3 $= \int \frac{2\, du}{u(1+u)}$ $\left(x = u^2,\ dx = 2u\, du \right)$

3 $= \int \left(\frac{2}{u} - \frac{2}{1+u} \right) du$ $\left(\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \Rightarrow 2 = (1+u)A + Bu \right.$
$\left. \Rightarrow A = 2,\ B = -2 \right)$

$= 2\ln|u| - 2\ln|1+u| + C$

$= 2\ln\sqrt{x} - 2\ln|1+\sqrt{x}| + C$

2 2

$\boxed{\ln|x| - 2\ln|1+\sqrt{x}| + C}$

or

(−2 pts if answer is left in terms of u) $\boxed{10}$

(8) 5. $\int \frac{x^2 + 5}{x^2 + 4}\, dx$

3 $= \int \left(1 + \frac{1}{x^2 + 4} \right) dx$

$= x + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

2 3

$\boxed{x + \frac{1}{2}\tan^{-1}\frac{x}{2} + C}$

$\boxed{8}$

(9) 6. Determine the constants in the partial fraction expansion

$$\frac{5x^2 - x + 3}{x(1+x^2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

$$5x^2 - x + 3 = (Ax+B)x + C(x^2+1)$$
$$= (A+C)x^2 + Bx + C$$

$$\Rightarrow \begin{cases} 5 = A + C \\ -1 = B \\ 3 = C \end{cases} \Rightarrow A = 5 - C = 2$$

3 3 3

$\boxed{A = 2 \quad, B = -1 \quad, C = 3}$

9

(16) 7. Determine whether each integral is convergent or divergent. Find its value if it is convergent. Important: You must show clearly how limits are involved.

a) $\displaystyle\int_0^{\infty} \frac{x}{(x^2+2)^2}\,dx \overset{3}{=} \lim_{t\to\infty} \int_0^t \frac{x}{(x^2+2)^2}\,dx$

$$= \lim_{t\to\infty} \int_2^{t^2+2} \frac{1}{2}\frac{du}{u^2} \quad (u = x^2+2,\ du = 2x\,dx)$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[-\frac{1}{u}\right]_2^{t^2+2} = \frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{2} - \frac{1}{t^2+2}\right] = \boxed{\frac{1}{4}}$$

3 2

8

b) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \overset{3}{=} \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t$$

$$= \lim_{t\to 1^-} \sin^{-1}t = \frac{\pi}{2}$$

3

$\boxed{\frac{\pi}{2}}$ 2

8

(7) 8. Set up but do not evaluate an integral for the length of the curve $y = \cos x, 0 \le x \le 2\pi$

* 0 credit if more than 1 item is wrong (limits : 1 item)

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1 + (\sin x)^2}\,dx$$

7

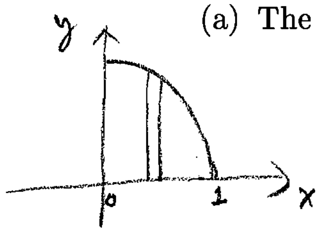
$\boxed{L = \int_0^{2\pi} \sqrt{1+\sin^2 x}\,dx}$ *

1 4 1

7

(12) 9. Consider the lamina in the first quadrant bounded by the curves $y = 1 - x^2$, $x = 0$ and $y = 0$, and with density $\rho = 1$. Find the following:

(a) The mass m of the lamina



NPC

$$m = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$\boxed{m = \frac{2}{3}}$ 3

(b) The moment M_y of the lamina about the y–axis

$$M_y = \int_0^1 x(1-x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$\boxed{M_y = \frac{1}{4}}$ 3

(c) The moment M_x of the lamina about the x–axis

$$M_x = \int_0^1 \frac{1}{2}\left[(1-x^2)^2 - 0\right] dx = \frac{1}{2}\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$\boxed{M_x = \frac{4}{15}}$ 3

(d) The center of mass $(\overline{x}, \overline{y})$ of the lamina

$$\overline{x} = \frac{M_y}{m} = \frac{3}{8}$$

$$\overline{y} = \frac{M_x}{m} = \frac{2}{5}$$

ok if consistent with above

$\boxed{(\overline{x}, \overline{y}) = \left(\frac{3}{8}, \frac{2}{5}\right)}$ 3

$\boxed{12}$

(10) 10. Determine whether the sequence converges or diverges. If it converges, find the limit. (You need not show work for this problem.)

NPC

(a) $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+\frac{1}{n}}{3-\frac{1}{n}} \longrightarrow \dfrac{1}{3}$ as $n \to \infty$

$\boxed{\frac{1}{3}}$ 2

(b) $\left\{\cos\dfrac{n\pi}{4}\right\}$ $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \ldots$

$\boxed{\text{diverges}}$ 2

(c) $\left\{\dfrac{(2n-1)!}{(2n+2)!}\right\}$ $\dfrac{(2n-1)!}{(2n+2)!} = \dfrac{1}{(2n+2)(2n+1)} \longrightarrow 0$ as $n \to \infty$

$\boxed{0}$ 2

(d) $\{2^{-n}\}$ $\lim\limits_{n\to\infty} 2^{-n} = \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, since $-1 < \frac{1}{2} < 1$

$\boxed{0}$ 2

(e) $a_n = n \sin\left(\dfrac{1}{n}\right)$

$\boxed{1}$ 2

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 1$$

$\boxed{10}$