MA166 — EXAM III — FALL 2018 — NOVEMBER 16, 2018
TEST NUMBER 11

INSTRUCTIONS:

1. **Do not open the exam booklet until you are instructed to do so.**
2. **Before you open the booklet fill in the information below and use a # 2 pencil to fill in the required information on the scantron.**
3. **MARK YOUR TEST NUMBER ON YOUR SCANTRON**
4. Once you are allowed to open the exam, make sure you have a complete test. There are 6 different test pages (including this cover page).
5. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers on this test booklet.
6. The test has 11 problems, worth 9 points; each everyone gets 1 point. The maximum possible score is 100 points. No partial credit.
7. Do not leave the exam room during the first 20 minutes of the exam.
8. After you have finished the exam, hand in your scantron and your test booklet to your recitation instructor.

DON'T BE A CHEATER:

1. Do not give, seek or obtain any kind of help from anyone to answer questions on this exam. If you have doubts, consult only your instructor.
2. Do not look at the exam or scantron of another student.
3. Do not allow other students to look at your exam or your scantron.
4. You may not compare answers with anyone else or consult another student until after you have finished your exam, given it to your instructor and left the room.
5. Do not consult notes or books.
6. **Do not handle** phones or cameras, calculators or any electronic device until after you have finished your exam, given it to your instructor and left the room.
7. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs collect the scantrons and the exams.
8. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above statements regarding academic dishonesty:

STUDENT NAME: SOLUTIONS

STUDENT SIGNATURE: ______________________

STUDENT ID NUMBER: ______________________

SECTION NUMBER AND RECITATION INSTRUCTOR: ______________________

1. Which of the following statements are true?

I. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges FALSE: This series diverges by the integral test

II. The series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges TRUE: This is an alternating series.

III. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges TRUE: This converges by the integral test.

A. I, II and III are true

B. I is true, II and III are false

C. I and II are true, III is false

D. I and III are true, II is false

E. II and III are true, I is false

2. Which of the following statements are true?

I. If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} |a_n| = 0$ TRUE: This is the first divergence test

II. If $\lim_{n\to\infty} |a_n| = 0$, then $\sum_{n=1}^{\infty} a_n$ always converges FALSE: Example $a_n = 1/n$

$\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} 1/n$ diverges

III. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

A. I, II and III are true

B. I is true, II and III are false

C. I and II are true, III is false

D. I and III are true, II is false

E. II and III are true, I is false

~~[illegible]~~ TRUE: If $\sum |a_n|$ converges, the series converges absolutely and so $\sum a_n$ converges.

Notice: $\sum a_n$ may converge and $\sum |a_n|$ diverges Ex: $\sum (-1)^n \frac{1}{n}$ converges, $\sum \frac{1}{n}$ diverges.

3. Which of the statements are true?

I. If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} n\, a_n$ diverges

TRUE. $n\,a_n > a_n$. By the comparison test $\sum n\,a_n$ diverges

II. If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges

TRUE. $\frac{a_n}{n} < a_n$. If $\sum a_n$ converges $\sum a_n$ converges

III. If $a_n > 0$ and $\sum_{n=2}^{\infty} (\ln n)^2 a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Also true because. $(\ln n)^2 a_n \geq a_n$. Comparison test $\sum a_n$ converges.

A. I and II are true, but III is false

B. I and III are true, but II is false

C. II and III are true, but I is false

D. I, II and III are true

E. I, II and III are false

4. Which of the statements are true?

I. If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sin(a_n)$ also converges

TRUE. $\lim_{n\to\infty} a_n = 0$ because the series converges. So $\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$. If $\sum a_n$ converges $\sum \sin(a_n)$ converges.

II. If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} e^{a_n}$ converges

FALSE. $\lim_{n\to\infty} a_n = 0$. $e^{a_n} \to 1$. The series $\sum e^{a_n}$ diverges

III. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \ln(a_n)$ diverges

TRUE. $\sum a_n$ converges so $\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} \ln(a_n) = -\infty$

Not zero, so the series diverges.

A. I and II are true, but III is false

B. I and III are true, but II is false

C. II and III are true, but I is false

D. I, II and III are true

E. I, II and III are false

5. Find all values of p such that the series $\sum_{k=1}^{\infty}\left(\frac{k^4+3k}{k^p+2}\right)^{1/3}$ converges.

A. $p > 8$

B. $p > 6$

C. $p > 5$

D. $p > 4$

$\boxed{\text{E. } p > 7}$

For k large $\left(\frac{k^4+3k}{k^p+2}\right)^{1/3}$

$$\sim \left(\frac{1+3/k^3}{k^{p-4}+2/k^4}\right) \sim \frac{1}{k^{\frac{p-4}{3}}}$$

So we need $\frac{p-4}{3} > 1 \quad p > 7$

6. Which of the following alternating series converge?

I) $\sum_{n=1}^{\infty}(-1)^{n-1}(\ln(n+1)-\ln(n))$ $= \sum_{n=1}^{\infty}(-1)^{n-1}\ln\left(\frac{n+1}{n}\right)$.

Is a convergent alternating series.

II) $\sum_{n=1}^{\infty}(-1)^{n-1}\cos(\frac{1}{n^2})$

III) $\sum_{n=1}^{\infty}(-1)^{n-1}\sin(\frac{1}{n})$

$\lim_{n\to\infty} \cos(1/n^2) = 1$. Series diverges.

$\sum_{n=1}^{\infty}(-1)^{n-1}\sin(1/n)$

is a convergent alternating series.

A. I, II and III

B. I and II only

$\boxed{\text{C. I and III only}}$

D. II and III only

E. None of them

7. Let $S = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^4}$ and its partial sum $S_n = \sum_{m=1}^{n}(-1)^{m-1}\frac{1}{m^4}$. According to the alternating series estimation theorem, what is the smallest n such that $|S - S_n| < 4^4 \times 10^{-8}$?

$\boxed{\text{A. } n = 25}$

B. $n = 24$

C. $n = 30$

D. $n = 35$

E. $n = 45$

Test says that

$$|S - S_n| \le b_{n+1} = \frac{1}{(n+1)^4} < \frac{4^4}{10^8}$$

$$(n+1)^4 > \frac{10^8}{4^4}$$

$$n+1 > \frac{100}{4} = 25$$

$$n > 24$$

$$\boxed{n = 25}$$

8. Let a_n, be a sequence defined recursively by $a_{n+1} = (-1)^{n-1}\left(\sqrt{n^2+2n} - \sqrt{n^2+n}\right)a_n$ and $a_1 \neq 0$. Which of the following is true?

$\boxed{\text{A. } \sum_{n=1}^{\infty} a_n \text{ converges absolutely}}$

B. $\sum_{n=1}^{\infty} a_n$ converges conditionally

C. $\sum_{n=1}^{\infty} a_n$ diverges

D. $\sum_{n=1}^{\infty} a_n$ could converge or diverge; it depends on a_1.

E. None of the above

$$\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{n^2+2n} - \sqrt{n^2+n}$$

$$= \left(\sqrt{n^2+2n} - \sqrt{n^2+n}\right)\frac{\left(\sqrt{n^2+2n} + \sqrt{n^2+n}\right)}{\sqrt{n^2+2n} + \sqrt{n^2+n}}$$

$$= \frac{n^2+2n-n^2-n}{\sqrt{n^2+2n} + \sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+2n} + \sqrt{n^2+n}}$$

$$= \frac{n}{n\sqrt{1+2/n} + n\sqrt{1+1/n}} = \frac{1}{\sqrt{1+2/n} + \sqrt{1+1/n}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1.$$

9. The radius and interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{n}$ are

A. $R = 2$ and $[0, 4]$

B. $R = 2$ and $(0, 4]$

C. $R = 2$ and $(0, 4)$

$\boxed{\text{D. } R = 1 \text{ and } (1, 3]}$

E. $R = 1$ and $(1, 3)$

$$|a_n| = |(-1)^n| \left|\frac{(x-2)^n}{n}\right| \qquad x \neq 2.$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{n+1}\right| \cdot \frac{n}{|x-2|^n} = \frac{n}{n+1}|x-2|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2| < 1.$$

Radius 1. 1 2 3

End points:

$x = 3$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges, $x = 1$ $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

$(1, 3]$

10. Find the Taylor series representation of the function $f(x) = \frac{1}{1-x}$ centered at -4.

A. $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(x+4)^n$

B. $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x+4)^n$

$\boxed{\text{C. } \sum_{n=0}^{\infty}\frac{1}{5^{n+1}}(x+4)^n}$

D. $\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x+4)^n$

E. $\sum_{n=0}^{\infty}\frac{1}{5^n}(x+4)^n$

$$f(x) = \frac{1}{1-x} = \frac{1}{1-(x+4-4)}$$

$$= \frac{1}{5-(x+4)} = \frac{1}{5}\,\frac{1}{1-\frac{x+4}{5}}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\frac{(x+4)^n}{5^n}$$

$$= \sum_{n=0}^{\infty}\frac{(x+4)^n}{5^{n+1}}.$$

11. Find the Taylor series representation of the function $f(x) = \dfrac{1}{(2-x)^2}$ centered at 0.

$\boxed{\text{A. } f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}}\, x^n}$

B. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2^{n+2}}\, x^n$

C. $f(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}\, x^n$

D. $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+1}}\, x^n$

E. $f(x) = \sum_{n=0}^{\infty} \frac{n+2}{2^n}\, x^n$

ANSWER KEYS:
Exam 11: 1-E, 2-D, 3-D, 4-B, 5-E, 6-C, 7-A, 8-A, 9-D, 10-C,11-A
Exam 22: 1-B, 2-B, 3-B, 4-A, 5-A, 6-D, 7-E, 8-C, 9-B, 10-E,11-D
Exam 33: 1-A, 2-C, 3-A, 4-D, 5-D, 6-B, 7-D, 8-B, 9-E, 10-B, 11-E
Exam 44: 1-D, 2-A, 3-C, 4-E, 5-B, 6-E, 7-E, 8-D, 9-C, 10-A, 11-B

$$f(x) = \frac{1}{(2-x)^2} = \frac{d}{dx}\frac{1}{2-x}.$$

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}.$$

$$\frac{d}{dx}\frac{1}{2-x} = \sum_{n=1}^{\infty} n\,\frac{x^{n-1}}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty}(n+1)\frac{x^n}{2^{n+2}}.$$