14. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following statements are true for any series \( \sum_{n=1}^{\infty} a_n \) with positive terms?

(I) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) converges. Not true. \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

(II) If \( \lim_{n \to \infty} \frac{a_n}{(\frac{1}{n})} = 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. True. Compare with \( \sum_{n=1}^{\infty} \frac{1}{n} \) which diverges and use limit comp. test.

(III) If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} \), then \( \sum_{n=1}^{\infty} a_n \) converges. True. Ratio test.

A. II only  B. II and III only  C. I and III only  D. all  E. (I) none

(b) Which of the following series converge?

(I) \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \) Ratio test: \( \left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n^!}{2^n} = \frac{2}{n+1} \to 0 < 1 \) conv.

(II) \( \sum_{n=1}^{\infty} \frac{n^3 - 1}{n^4 - n} \) div. Compare with \( \sum_{n=1}^{\infty} \frac{1}{n} \) and use limit comp. test.

(III) \( 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots \) conv. by Alt. Series Test

A. I only  B. III only  C. II and III only  D. I and III only  E. none
2. Determine whether each series is convergent or divergent. You must show all necessary work and write your conclusion in the small box.

(a) \[ \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + 4}} \quad \text{For problems 2(a) and 2(b) look first for "conv" or "div"} \]

\[ \text{If wrong} \rightarrow \text{Opt for problem} \]
\[ \text{If right} \rightarrow \text{Check work and test} \]

Show all necessary work here:

\[ \text{Compare with} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \text{which converges} \]
\[ (p\text{-series}, \quad p = \frac{3}{2} > 1) \]

\[ \frac{n}{\sqrt{n^5 + 4}} < \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}} \quad \text{for all } n \]

\[ \text{Or} \quad \lim_{n \to \infty} \frac{n}{\sqrt{n^5 + 4}} = \lim_{n \to \infty} \frac{n^{5/2}}{n^{5/2} \left(1 + \frac{4}{n^2}\right)} = 1 \]

By the \text{Comparison test, the series is \textbf{convergent}} [or \text{limit comparison}]

(b) \[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1} \]

Show all necessary work here:

\[ \text{Alternating Series Test} \quad b_n = \frac{\sqrt{n}}{n+1} \]

(i) \[ b_n \text{ decreasing?} \]

Let \[ f(x) = \frac{\sqrt{x}}{x+1} \]

\[ f'(x) = \frac{(x+1)^{-3/2} - \sqrt{x}}{(x+1)^2} = \frac{x+1 - 2\sqrt{x}}{2\sqrt{x} (x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} < 0 \]

\[ \therefore \text{Yes } b_n \text{ is decreasing} \]

(ii) \[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \to \infty} \frac{1}{n^{1/2} + \frac{1}{\sqrt{n}}} = 0 \]

By the \text{Alternating series test, the series is \textbf{convergent}}
3. Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} \) is convergent or divergent. If it is convergent, find its sum.

\[
\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} + \frac{(-3)}{4^2} + \frac{(-3)^2}{4^3} + \frac{(-3)^3}{4^4} + \cdots = \frac{1}{4} \left[ 1 + (-\frac{3}{4}) + (-\frac{3}{4})^2 + (-\frac{3}{4})^3 + \cdots \right] \text{ geometric series}
\]

\[
= \frac{1}{4} \cdot \frac{1}{1 - (-\frac{3}{4})} = \frac{1}{4 + 3} = \frac{1}{7}
\]

If answer is wrong, give 5 points for attempting to sum.

4. Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \).

(a) Write out the first six terms of the series. 

\[
1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \cdots
\]

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

\[
\frac{1}{5^3} = \frac{1}{125} < \frac{1}{100} = 0.01 \quad \text{3 terms}
\]

\[
\frac{1}{4^3} = \frac{1}{64} > 0.01
\]

5. Determine whether the series \( \sum_{n=1}^{\infty} \frac{\cos \left( \frac{n\pi}{2} \right)}{n^2 + 4n} \) is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

Absolutely convergent?

\[
\sum_{n=1}^{\infty} \left| \frac{\cos \left( \frac{n\pi}{2} \right)}{n^2 + 4n} \right| \quad \text{conv? Compare with} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

which is conv. (p-series, p = 2 > 1)

\[
\left| \frac{\cos \left( \frac{n\pi}{2} \right)}{n^2 + 4n} \right| = \left| \frac{\cos \left( \frac{n\pi}{2} \right)}{n^2 + 4n} \right| < \frac{1}{n^2} \quad \text{for all} \ n
\]

The series is absolutely convergent by the comparison test

Opt. for problem if answer is wrong

The series is absolutely convergent
6. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n x^n}{2^n}$. Don’t forget to test for convergence at the end points of the interval. You must show all work.

$$a_n = \frac{n x^n}{2^n}$$

Ratio test: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1) x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n x^n} \right| = \lim_{n \to \infty} \left| \frac{n+1}{2} \cdot \frac{x}{n} \right| = \frac{|x|}{2}$

$$\therefore \text{series converges if } \frac{|x|}{2} < 1, \text{ or } |x| < 2, \text{ or } -2 < x < 2.$$

When $x = -2$: $\sum_{n=1}^{\infty} \frac{n (-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$ D.N.E.

When $x = 2$: $\sum_{n=1}^{\infty} \frac{n 2^n}{2^n} = \sum_{n=1}^{\infty} n$ D.N.E. by the test for diverg. $\lim_{n \to \infty} \frac{n}{n+1} = 1$

\[ \text{Interval of convergence: } (-2, 2) \]

7. Find a power series representation for $f(x) = \frac{1}{1 + 9x^2}$ and determine the radius of convergence $R$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1$$

Replace $x$ by $-9x^2$:

$$\frac{1}{1 + 9x^2} = \sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=1}^{\infty} (-1)^n 9^n x^{2n}$$

which conv. $\lim_{n \to \infty} |(-9x^2)^n| < 1$

or $|x| < \frac{1}{3}$

8. Find the Taylor series for $f(x) = 1 + x + x^2$ about $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = f(2) + \frac{f^{(1)}(2)}{1!} (x-2) + \frac{f^{(2)}(2)}{2!} (x-2)^2 + \ldots$$

$f(2) = 7$

$f^{(1)}(2) = 5$

$f^{(2)}(2) = 2$

$f^{(n)}(2) = 0$ for all $n \geq 3$

\[ \therefore \ f(x) = 7 + \frac{5}{1!} (x-2) + \frac{2}{2!} (x-2)^2 \]

-2 pts for each additional term

\[ f(x) = 7 + 5 (x-2) + (x-2)^2 \]