NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

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(14) 1. Circle the letter of the correct response. (You need not show work for this problem). NPC

(a) Which of the following statements are true for any series $\sum_{n=1}^{\infty} a_n$ with positive terms?

(I) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. Not true. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(II) If $\lim_{n\to\infty} \frac{a_n}{\left(\frac{1}{n}\right)} = 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. True. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which div. and use limit comp. test.

(III) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n$ converges. True. Ratio test.

A. II only B. II and III only C. I and III only D. all E. (I) none

$\boxed{7}$

(b) Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \frac{2}{n+1} \xrightarrow[\text{as } n\to\infty]{} 0 < 1$ $\therefore$ conv.

(II) $\sum_{n=1}^{\infty} \frac{n^3-1}{n^4-n}$ div. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ and use Limit Comp. Test

(III) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots$ conv. by Alt. Series Test

A. I only B. III only C. II and III only D. I and III only E. none

$\boxed{7}$

(20) 2. Determine whether each series is convergent or divergent. You must show all necessary work and write your conclusion in the small box.

For problems 2(a) and 2(b) look first for "conv" or "div"
If wrong → 0 pts for problem
If right → check work and test

~~8~~ (a) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$

Show all necessary work here:

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ 3 which converges 1 (p-series, $p = \frac{3}{2} > 1$) 1

$\frac{n}{\sqrt{n^5+4}} < \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$ 2 for all n.

[Or $\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^5+4}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{5/2}}{n^{5/2}\sqrt{1+\frac{4}{n^5}}} = 1$]

3

By the Comparison test, the series is convergent $\boxed{10}$

[or limit comparison]

~~8~~ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

Show all necessary work here:

Alternating Series Test. $b_n = \frac{\sqrt{n}}{n+1}$. 2

(i) b_n decreasing?

Let $f(x) = \frac{\sqrt{x}}{x+1}$

$f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}\,(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} < 0$ 1 for $x > 1$

∴ Yes b_n is decreasing 2

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$ 2

3

By the alternating series test, the series is convergent $\boxed{10}$

(10) 3. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} + \frac{(-3)}{4^2} + \frac{(-3)^2}{4^3} + \frac{(-3)^3}{4^4} + \cdots$$

$$= \frac{1}{4}\left[1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right]$$ geometric series

$$= \frac{1}{4} \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{4+3} = \frac{1}{7}$$

$\boxed{\frac{1}{7}}$ 10

If answer is wrong, give 5 points for attempting to sum using geometric series.

(10) 4. Consider the series $\sum_{n=\cancel{0}1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$.

(a) Write out the first six terms of the series. −1 pt if only 1 term is wrong

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \cdots$$ or 4 or

$$1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \frac{1}{216} + \cdots$$

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error < 0.01.

3

$$\frac{1}{5^3} = \frac{1}{125} < \frac{1}{100} = 0.01$$ 3

$$\frac{1}{4^3} = \frac{1}{64} > 0.01$$

$\boxed{4 \text{ terms}}$

(10) ~~(8)~~ 5. Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos(\frac{n}{2})}{n^2+4n}$ is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

Absolutely convergent?

$$\sum_{n=1}^{\infty} \left|\frac{\cos(\frac{n}{2})}{n^2+4n}\right| \text{ conv?}$$ 3

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 2

which is conv. (p-series, p=2>1) 1

$$\left|\frac{\cos(\frac{n}{2})}{n^2+4n}\right| = \frac{|\cos(\frac{n}{2})|}{n^2+4n} < \frac{1}{n^2}$$ 2 for all n

∴ The series is absolutely convergent by the comparison test 2

2 pts for correct answer but wrong proof.

0 pts for problem if answer is wrong

$\boxed{\text{The series is absolutely convergent}}$ 10

(*) −3pts if | | (abs. value sign) is missing here and appears later on

0 pts beyond this point if this is wrong

(16) 6. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n\,x^n}{2^n}$. Don't forget to test for convergence at the end points of the interval. You must show all work.

$a_n = \frac{n x^n}{2^n}$

Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{2^{n+1}}}{\frac{n x^n}{2^n}}\right| = \lim_{n\to\infty}\frac{n+1}{2n}|x| = \frac{|x|}{2}$ (6) (*) (2)

∴ series converges if $\frac{|x|}{2} < 1$, or $|x| < 2$, or $-2 < x < 2$. (2)

When $x = -2$: $\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$ div. (2) by the test for div. ($\lim_{n\to\infty}(-1)^n n$ DNE)

When $x = 2$: $\sum_{n=1}^{\infty} \frac{n\,2^n}{2^n} = \sum_{n=1}^{\infty} n$ div. (2) by the test for div. ($\lim_{n\to\infty} n = \infty$)

∴ interval of convergence: $(-2, 2)$

$\boxed{(-2, 2) \text{ or } -2 < x < 2}$ (1) (1)

16

(10) 7. Find a power series representation for $f(x) = \frac{1}{1+9x^2}$ and determine the radius of convergence R.

NPC

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$

Replace x by $-9x^2$:

$\frac{1}{1+9x^2} = \sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=1} (-1)^n 9^n x^{2n}$

which conv. for $|-9x^2| < 1$

or $|x| < \frac{1}{3}$

$\frac{1}{1+9x^2} = \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}$ (1) (1) (4)

$R = \frac{1}{3}$ (4)

10

(10) 8. Find the Taylor series for $f(x) = 1 + x + x^2$ ~~about~~ centered at $a = 2$.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$ (4) or $= f(2) + \frac{f^{(1)}(2)}{1!}(x-2) + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \cdots$

$f(x) = 1 + x + x^2$ $\quad f(2) = 7$

$f^{(1)}(x) = 1 + 2x$ $\quad f^{(1)}(2) = 5$

$f^{(2)}(x) = 2$ $\quad f^{(2)}(2) = 2$

$f^{(n)}(x) = 0$ $\quad f^{(n)}(2) = 0$ for all $n \geq 3$

∴ $f(x) = 7 + \frac{5}{1!}(x-2) + \frac{2}{2!}(x-2)^2$

−2 pts for each additional term

$f(x) = 7 + 5(x-2) + (x-2)^2$ (2) (2) (2)

10