DIRECTIONS

1. Write your name, student ID number, recitation instructor’s name and recitation time in the space provided above. Also write your name at the top of pages 2, 3, and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this test.

(14) 1. Circle the letter of the correct response. (You need not show work for this problem).

(a) Which of the following statements are true for any series \( \sum_{n=1}^{\infty} a_n \)?

(I) If \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent, then \( \sum_{n=1}^{\infty} \frac{1}{n} \) is convergent. \( \text{True, by theorem} \)

(II) If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \sum_{n=1}^{\infty} a_n \) is convergent. \( \text{Not true, ex: } \sum_{n=1}^{\infty} \frac{1}{n^2} \)

(III) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), then \( \sum_{n=1}^{\infty} a_n \) is divergent. \( \text{Not true, ex: } \sum_{n=1}^{\infty} \frac{1}{n^2} \)

\( A \) (I) only  \( B \) (I) and (II) only  \( C \) (II) only  \( D \) (III) only  \( E \) none  

(b) Which of the following series converge?

(I) \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \), with any \( p > 0 \). \( \text{Converges, by alternating series test} \)

(II) \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{3n^3 - 2n} \) Diverges. Comparison test, compare with \( \sum_{n=1}^{\infty} \frac{1}{3n} \)

(III) \( \sum_{n=1}^{\infty} \frac{\sin(2n)}{n^3} \) Converges because it converges absolutely. Compare \( \sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{n^3} \right| \) with \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

\( A \) (I) only  \( B \) (II) only  \( C \) (III) only  \( D \) (I) and (III) only  \( E \) all  

(7)
2. Determine whether each series is convergent or divergent. You must verify that the conditions of the test are satisfied and write your conclusion in the small box.

For problems 2(a), 2(b), and 2(c) look first for \( \text{conv.} \) or \( \text{div.} \).

If wrong \( \to \) 0 pts for problem
If right \( \to \) check work and test

Show all necessary work here:

Integral test. Let \( f(x) = \frac{1}{x \ln x} \quad x \in [2, \infty) \)

\( f \) is continuous, positive and decreasing for \( x \in [2, \infty) \)

and \( f(n) = \frac{1}{n \ln n} = a_n \)

\[
\int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \left[ \ln (\ln x) \right]_2^t
\]

\[
= \lim_{t \to \infty} \left[ \ln (\ln t) - \ln (\ln 2) \right] = \infty
\]

\[
\therefore \int_2^\infty \frac{1}{x \ln x} \, dx \quad \text{is divergent}
\]

By the integral test, the series is \text{divergent}.

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(8) (b) \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4} \)

Show all necessary work here:

**Alternating series test** \( b_n = \frac{\sqrt{n}}{n+4} \)

(i) \( b_n \) decreasing? Let \( f(x) = \frac{\sqrt{x}}{x+4} \)

\[
f'(x) = \frac{\frac{1}{2 \sqrt{x}} - \sqrt{x}}{(x+4)^2} = \frac{x+4 - 2x}{2 \sqrt{x} (x+4)^2} = \frac{4-x}{2 \sqrt{x} (x+4)^2}
\]

\( f'(x) < 0 \) for \( x > 4 \) \( \quad \leftarrow (2) \)

\( \therefore b_n \) are decreasing for \( n > 4 \)

(ii) \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\sqrt{n}}{n+4} = 0 \) \( \quad \leftarrow (2) \)

By the alternating series test, the series is \text{convergent}. \( \quad \leftarrow (2) \)
(8) (c) \( \sum_{n=1}^{\infty} n \sin \left( \frac{1}{n} \right) \)

Justify your answer and show all necessary work here:

\[
\alpha_n = n \sin \left( \frac{1}{n} \right)
\]

\[
\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0
\]

\[
\therefore \text{ series is divergent by theorem (or "test for divergence")}
\]

The series is **divergent**

(8) 3. Find the sum of the series if it is convergent or write "divergent" in the box. No partial credit.

(a) \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n} \) = \( \frac{1}{4} \sum_{n=1}^{\infty} \left( -\frac{1}{4} \right)^{n-1} = \frac{1}{4} \left( 1 + \left( -\frac{1}{4} \right) + \left( -\frac{1}{4} \right)^2 + \cdots \right) \)

\[
\text{geometric series}
\]

\[
\frac{1}{1 - (-\frac{1}{4})} = \frac{4}{3} \]

(b) \( \sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{3^n} \) = \( \frac{1}{3} \sum_{n=1}^{\infty} \left( -\frac{4}{3} \right)^{n-1} \)

\[
\text{geometric series}
\]

divergent because \( \left| -\frac{4}{3} \right| > 1 \)

(8) 4. Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \).

(a) Write out the first five terms of the series.

\[
1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \quad \text{or 4}
\]

(b) Find the smallest number of terms that we need to add in order to estimate the sum of the series with error \( < \frac{1}{20} \)

4 NPCs 3 terms

(10) 5. Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \) is absolutely convergent, conditionally convergent, or divergent. You must justify your answer.

Absolutely convergent? Use ratio test

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^n} \right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \rightarrow 0 < 1 \quad \text{as} \quad n \rightarrow \infty
\]

The series is **absolutely convergent**
(16) 6. For the power series \( \sum_{n=1}^{\infty} (-1)^n 4^n x^n \), find the following, showing all work.

(a) The radius of convergence \( R \).

The ratio test:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{4|x|}{n} \xrightarrow{n \to \infty} 4|x|
\]

Series converges if \( 4|x| < 1 \) or \( |x| < \frac{1}{4} \).

\( R = \frac{1}{4} \)

(b) The interval of convergence. (Don't forget to check the end points).

Series converges if \( -\frac{1}{4} < x < \frac{1}{4} \).

When \( x = -\frac{1}{4} \):
\[
\sum_{n=1}^{\infty} (-1)^n 4^n \left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} n
\]
diverges.

When \( x = \frac{1}{4} \):
\[
\sum_{n=1}^{\infty} (-1)^n 4^n \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} (4)^n
\]
diverges.

Intervals of convergence:
\((-\frac{1}{4}, \frac{1}{4})\)

or \(-\frac{1}{4} < x < \frac{1}{4}\).

(10) 7. (a) Find the power series representation of \(-\frac{1}{5-x}\) (about \( a = 0 \)).

\(-\frac{1}{5-x} = -\frac{1}{5} \cdot \frac{1}{1-x/5} = -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n
\]

\(-\frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}
\]

For \( x = 0 \), \( \ln 5 = C \).

(b) Use integration of power series and the fact that \( \ln(5-x) = -\int \frac{1}{5-x} dx \) for a certain value of the constant of integration \( C \), to find the power series for \( \ln(5-x) \).

\( \ln(5-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}(n+1)} + C
\]

Set \( x = 0 \) : \( \ln 5 = C \).

(10) 8. Find the Taylor series for \( f(x) = \frac{1}{x} \) centered at \( a = 2 \).

\( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n \)

\( f(2) = 2^{-1} \)

\( 3 \)

or \( \frac{1}{x} = \frac{1}{2} + (x-2) + \frac{(x-2)^2}{2}
\]

\( = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(-x/2)^n}{n!}
\]

\( \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1} (x-2)^n}
\)