NAME SOLUTIONS

STUDENT ID # ____________

RECITATION INSTRUCTOR ____________

RECITATION TIME ____________

LECTURER ____________

INSTRUCTIONS

1. There are 10 different test pages (including this cover page). Make sure you have a complete test.
2. Fill in the above items in print. I.D.# is your 9 digit ID (probably your social security number). Also write your name at the top of pages 2–10.
3. Do any necessary work for each problem on the space provided or on the back of the pages of this test booklet. Circle your answers in this test booklet. No partial credit will be given, but if you show your work on the test booklet, it may be used in borderline cases.
4. No books, notes or calculators may be used on this exam.
5. Each problem is worth 8 points. The maximum possible score is 200 points.
6. Using a #2 pencil, fill in each of the following items on your answer sheet:
   (a) On the top left side, write your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION, write in your division and section number and fill in the little circles. (For example, for division 9 section 1, write 0901. For example, for division 38 section 2, write 3802).
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student ID number, and fill in the little circles.
   (d) Using a #2 pencil, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect. Use a #2 pencil.
7. After you have finished the exam, hand in your answer sheet and your test booklet to your recitation instructor.

1. Consider the triangle ABC with vertices $A = (0,0,0)$, $B = \left(x, \frac{x}{2}, 0\right)$, and $C = (1,2,0)$, where $x > 0$. If the area of the triangle is 6, then $x =$

$\overrightarrow{AB} = x\vec{i} + \frac{x}{2}\vec{j}$

$\overrightarrow{AC} = \vec{i} + 2\vec{j}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & \frac{x}{2} & 0 \\ 1 & 2 & 0 \end{vmatrix} = \left(2x - \frac{x}{2}\right)\vec{k} = \frac{3x}{2}\vec{k}$$

Area of $ABC = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\left|\frac{3x}{2}\right| = \frac{3x}{4}$

$\frac{3x}{4} = 6 \longrightarrow x = 8$

A. 4
B. 8
C. 3
D. 5
E. 2

2. Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + x\vec{j}$, where $x > 0$. Find x so that $\|pr_{\vec{a}}\vec{b}\| = 1$.

$$pr_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|a\|^2}\vec{a} = \frac{1+x}{3}(\vec{i} + \vec{j} + \vec{k})$$

$$\|pr_{\vec{a}}\vec{b}\| = \left|\frac{1+x}{3}\right|\sqrt{3} = \frac{1+x}{\sqrt{3}}$$

$\frac{1+x}{\sqrt{3}} = 1 \longrightarrow x = \sqrt{3} - 1$

A. 2
B. $\sqrt{2} - 1$
C. 3
D. $\sqrt{3} - 1$
E. $\sqrt{3}$

3. $\lim_{x\to 0} \frac{x \sin x}{1 - \cos^3 x} \overset{L'H}{=} \lim_{x\to 0} \frac{x\cos x + \sin x}{-3\cos^2 x(-\sin x)}$

$\frac{0}{0}$

$$= \lim_{x\to 0} \frac{x\cos x + \sin x}{3\sin x \cos^2 x}$$

$\frac{0}{0}$

$$= \lim_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{-6\sin x\cos x\sin x + 3\cos^3 x}$$

$$= \frac{2}{3}$$

A. $\frac{2}{3}$
B. $-\frac{2}{3}$
C. 0
D. 1
E. does not exist

4. $\int \tan x \sec^3 x dx =$ $\int \sec^2 x (\sec x \tan x) dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

$= \int u^2 du = \frac{u^3}{3} + C = \frac{1}{3}\sec^3 x + C$

A. $\frac{1}{2}\tan^2 x + C$

B. $\frac{1}{3}\sec^3 x + C$

C. $\frac{1}{4}\sec^4 x + C$

D. $\frac{1}{3}\tan^3 x + C$

E. $\frac{1}{3}\csc^3 x + C$

5. $\int_0^{\frac{\pi}{3}} \sin x \sec^2 x \, dx =$ $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x} dx = -\int_1^{\frac{1}{2}} \frac{du}{u^2} =$

$u = \cos x \quad du = -\sin x \, dx$

$x = 0 \rightarrow u = 1$

$x = \frac{\pi}{3} \rightarrow u = \frac{1}{2}$

$= \frac{1}{u}\Big|_1^{1/2} = \frac{1}{\frac{1}{2}} - 1 = 1$

A. 2

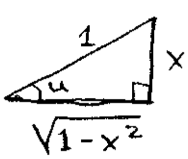
B. 1

C. 0

D. $\frac{1}{2}$

E. 3

6. The integral $\int_0^1 x^2\sqrt{1-x^2}\, dx$ is transformed by a trigonometric substitution to



$x = \sin u \quad dx = \cos u \, du$

$\sqrt{1-x^2} = \cos u$

$x = 0 \rightarrow u = 0$

$x = 1 \rightarrow u = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \sin^2 u \cos u \cos u \, du$

A. $\int_0^1 \sin^2 u \cos^2 u \, du$

B. $\int_0^{\frac{\pi}{4}} \sin^2 u \cos^3 u \, du$

C. $\int_{\frac{\pi}{2}}^0 \sin^2 u \cos^3 u \, du$

D. $\int_0^{\frac{\pi}{2}} \sin^3 u \cos^2 u \, du$

E. $\int_0^{\frac{\pi}{2}} \sin^2 u \cos^2 u \, du$

7. $\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x^2}\,dx$

$= \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$

$= \lim_{b\to\infty} \left(\tan^{-1}b - \tan^{-1}1\right)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

A. $\frac{\pi}{2}$

B. integral diverges

C. π

D. $\frac{\pi}{3}$

E. $\frac{\pi}{4}$

8. $\int_1^2 x\ln x\,dx = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}\,dx =$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$= 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2$

$= 2\ln 2 - \left[1 - \frac{1}{4}\right]$

$= 2\ln 2 - \frac{3}{4}$

A. $2\ln 2 - \frac{3}{4}$

B. $\ln 2 - \frac{3}{4}$

C. $\frac{1}{2}\ln 2 - \frac{1}{4}$

D. $\frac{3}{4}$

E. $2\ln 2 - \frac{1}{2}$

9. $\int_2^3 \frac{1}{x^2-1}\,dx =$

$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$1 = Ax - A + Bx + B.$

$A + B = 0 \qquad B - A = 1$

$\therefore B = \frac{1}{2} \quad A = -\frac{1}{2}$

$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \left(-\frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}\right)dx$

$= \left[-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|\right]_2^3 = -\frac{1}{2}\ln 4$

$+ \frac{1}{2}\ln 2 + \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\frac{2\cdot 3}{4} = \frac{1}{2}\ln\frac{3}{2}$

A. $\frac{1}{2}\ln 3$

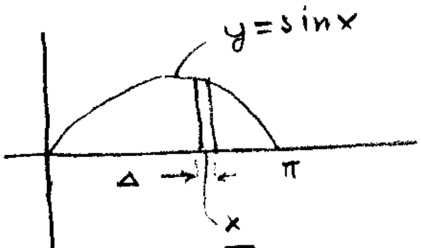
B. $\ln\frac{3}{2}$

C. $\ln\frac{1}{2}$

D. $\frac{1}{2}\ln\frac{3}{2}$

E. $\ln 3$

10. Let R be the region between the graph of $y = \sin x$ and the x-axis on the interval $[0, \pi]$. The volume of the solid obtained by revolving R about the x-axis is



$$\Delta V = \pi \sin^2 x \, \Delta x$$

$$V = \int_0^{\pi} \pi \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1-\cos 2x}{2} \, dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\pi} = \pi \left[\frac{1}{2}\pi \right] = \frac{\pi^2}{2}$$

A. $\frac{\pi}{2}$
B. π^2
C. π
D. $\frac{\pi^2}{2}$
E. $\frac{\pi^2}{3}$

11. The length of the graph of $f(x) = 5 + \frac{2}{3}x^{\frac{3}{2}}$, $0 \le x \le 2$, is equal to

$$L = \int_0^2 \sqrt{1 + [f'(x)]^2} \, dx$$

$$= \int_0^2 \sqrt{1+x} \, dx = \int_1^3 u^{1/2} \, du$$

$u = 1 + x \quad du = dx$
$x = 0 \rightarrow u = 1$
$x = 2 \rightarrow u = 3$

$$= \frac{2}{3} u^{3/2} \Big|_1^3 = \frac{2}{3}(3\sqrt{3} - 1)$$

A. $3\sqrt{3} - 1$
B. $\frac{2}{3}(3\sqrt{3} - 2)$
C. $\frac{2}{3}(3\sqrt{3} - 1)$
D. $2\sqrt{2} - 3$
E. $3\sqrt{2} - \sqrt{2}$

12. Suppose that a force of 4 lbs is required to stretch a spring 2 ft beyond its natural length. How much work is required to stretch it from 2 ft to 3 ft beyond its natural length?

$$F = kx \quad \rightarrow \quad 4 = k2 \quad \rightarrow \quad k = 2$$

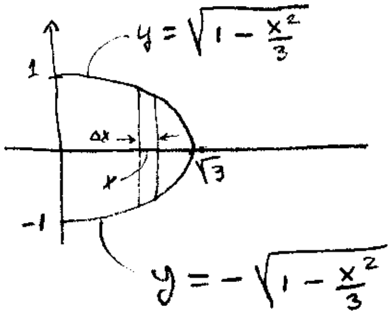
$$W = \int_2^3 2x \, dx = x^2 \Big|_2^3 = 9 - 4 = 5$$

A. 9 ft-lbs
B. 4 ft-lbs
C. 6 ft-lbs
D. 5 ft-lbs
E. 2 ft-lbs

13. Let R be the region bounded by the y axis and the graphs of

$$f(x) = \sqrt{1 - \frac{x^2}{3}} \text{ and } g(x) = -\sqrt{1 - \frac{x^2}{3}}, \text{ for } 0 \le x \le \sqrt{3}$$

Given that the area of R is $\sqrt{3}\pi$, find the center of gravity of R.



From symmetry $\bar{y} = 0$

$M_y = A\bar{x}$

$$\Delta M_y = x\left[\sqrt{1 - \frac{x^2}{3}} - \left(-\sqrt{1 - \frac{x^2}{3}}\right)\right]\Delta x$$

$$M_y = \int_0^{\sqrt{3}} 2x\sqrt{1 - \frac{x^2}{3}}\,dx = -3\int_1^0 u^{1/2}\,du =$$

$u = 1 - \frac{x^2}{3}$ $du = -\frac{2x}{3}dx$

$x = 0 \to u = 1$, $x = \sqrt{3} \to u = 0$

$$= -3 \cdot \frac{2}{3} u^{3/2}\Big|_1^0 = +2$$

$$2 = \sqrt{3}\pi\,\bar{x} \to \bar{x} = \frac{2\sqrt{3}}{3\pi}$$

A. $\left(\frac{2\sqrt{3}}{3\pi}, 0\right)$

B. $\left(\frac{\sqrt{3}\pi}{2}, 0\right)$

C. $\left(0, \frac{\sqrt{3}\pi}{2}\right)$

D. $\left(\frac{\sqrt{3}}{2}, 0\right)$

E. $\left(0, \frac{\sqrt{3}}{2}\right)$

14. $\lim_{k\to\infty}\left(1 + \frac{1}{3k}\right)^{2k} = \lim_{x\to\infty}\left(1 + \frac{1}{3x}\right)^{2x}$

$$= \lim_{x\to\infty} e^{2x\ln\left(1 + \frac{1}{3x}\right)}$$

$$= e^{\lim_{x\to\infty} 2x\ln\left(1 + \frac{1}{3x}\right)}$$

A. e^2

B. 1

C. ∞

D. $e^{\frac{2}{3}}$

E. e^6

$$\lim_{x\to\infty} 2x\ln\left(1 + \frac{1}{3x}\right)$$

$$= 2\lim_{x\to\infty}\frac{\ln\left(1 + \frac{1}{3x}\right)}{\frac{1}{x}} \overset{L'H}{=} 2\lim_{x\to\infty}\frac{\frac{1}{1 + \frac{1}{3x}}\left(-\frac{1}{3x^2}\right)}{-\frac{1}{x^2}} = \frac{2}{3}$$

$\frac{0}{0}$

$$\therefore \lim_{k\to\infty}\left(1 + \frac{1}{3k}\right)^{2k} = e^{2/3}$$

15. Which of the following series converge?

(I) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$, conv. (alt. ser. test)

(II) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$, div. $\left(\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0\right)$

(III) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ conv. (ratio test)

$$\left(\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1\right)$$

A. (II) and (III) only

B. (III) only

C. (I) and (II) only

D. (I) only

E. (I) and (III) only

16. Which of the following series converge absolutely?

(I) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, no $\left(\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}\right.$ harm. ser. div$\left.\right)$

(II) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$, yes $\left(\left|\frac{\cos n}{n^2}\right| < \frac{1}{n^2}\right.$, comp with $\left.\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$

(III) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^3+1}}$ yes

$$\left(\left|(-1)^n \frac{1}{\sqrt{n^3+1}}\right| = \frac{1}{(n^3+1)^{1/2}} < \frac{1}{n^{3/2}},\right.$$

comp. with $\left.\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\right)$

A. (II) and (III) only

B. (III) only

C. (I) and (II) only

D. (I) only

E. (I) and (III) only

17. The series $\sum_{n=1}^{\infty} 2\left(\frac{3^n}{7^{n+1}}\right) = \frac{2}{7} \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n$

$$= \frac{2}{7}\left[\frac{3}{7} + \left(\frac{3}{7}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots\right]$$

$$= \frac{2}{7} \cdot \frac{3}{7}\left[1 + \left(\frac{3}{7}\right) + \left(\frac{3}{7}\right)^2 + \cdots\right]$$

$$= \frac{2 \cdot 3}{7 \cdot 7} \frac{1}{1 - \frac{3}{7}} = \frac{2 \cdot 3}{7 \cdot 7} \frac{1}{\frac{4}{7}}$$

$$= \frac{3}{14}$$

A. diverges

B. $= \frac{3}{14}$

C. $= \frac{1}{2}$

D. $= \frac{3}{7}$

E. $= \frac{4}{7}$

18. The series $\sum_{n=2}^{\infty} \frac{1}{\ln n + 3^n}$

A. converges by comparison with $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

B. diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

C. converges by comparison with $\sum_{n=2}^{\infty} \frac{1}{3^n}$

D. diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{3^n}$

E. diverges by the ratio test

19. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{n!} x^n$ is

gen. rat. test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2 x^{n+1}}{(n+1)!}}{\frac{n^2 x^n}{n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{1}{n+1} |x| = 0$$

< 1 for all x

A. e

B. $\frac{1}{e}$

C. 1

D. ∞

E. 2

20. Use the Taylor series of e^{-x^3} to approximate $\int_0^1 e^{-x^3}\,dx$ with error less than 0.01. The smallest number of terms of the series that are needed for this accuracy is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\int_0^1 e^{-x^3} dx = \int_0^1 \left[1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \frac{x^{12}}{24} - \cdots\right] dx$$

$$= \left[x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + \frac{x^{13}}{24 \cdot 13} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} \Big| + \frac{1}{312} - \cdots$$

A. 2

B. 3

C. 4

D. 5

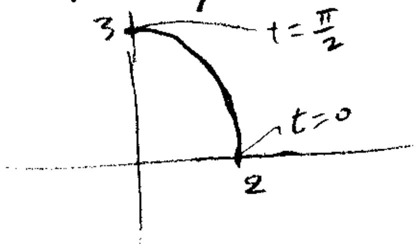
E. 6

21. The parametric equations of a curve C are:

$$x = 2\cos t, \;\; y = 3\sin t \text{ for } 0 \le t \le \frac{\pi}{2}$$

The curve C is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



A. a quarter of a circle

B. an ellipse

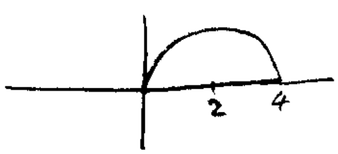
C. a half of an ellipse

D. a half of a circle

E. a quarter of an ellipse

22. The upper half ($y \ge 0$) of the circle $(x-2)^2 + y^2 = 4$ is described in polar coordinates by

$$x^2 - 4x + 4 + y^2 = 4$$
$$x^2 + y^2 = 4x$$
$$r^2 = 4r\cos\theta$$
$$r = 4\cos\theta$$



A. $r = 4\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$

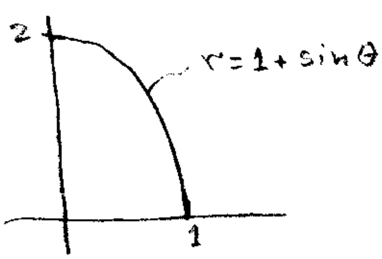
B. $r = 2\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$

C. $r = 2\cos\theta \quad 0 \le \theta \le \pi$

D. $r = 4\cos\theta \quad 0 \le \theta \le \pi$

E. $r = \cos\theta \quad 0 \le \theta \le \pi$

23. The area of the region in the first quadrant and inside the cardioid $r = 1 + \sin\theta$ is



$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2 \, d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + 2\sin\theta + \sin^2\theta)\, d\theta$$
$$= \frac{1}{2}\left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - \frac{1}{2}(-2)$$
$$= \frac{1}{2}\left(\frac{3\pi}{\ldots}\right)$$

A. $\left(\frac{3\pi}{4} + 2\right)$

B. $\frac{1}{2}\left(\frac{3\pi}{4} + 1\right)$

C. $\frac{1}{2}\left(\frac{3\pi}{4} + 2\right)$

D. $\left(\frac{\pi}{4} + 1\right)$

E. $\frac{1}{2}(\pi + 2)$

24. The length of the parametrized curve

$$x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2 + 3, \quad 0 \le t \le 1$$

is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{t^4 + t^2}\, dt$$

$$= \int_0^1 \sqrt{t^2+1}\, t\, dt = \int_1^2 \frac{1}{2} u^{1/2} du =$$

$u = t^2+1 \quad du = 2t\,dt$

$t = 0 \rightarrow u = 1$

$t = 1 \rightarrow u = 2$

$$= \frac{1}{2}\frac{2}{3} u^{3/2}\Big|_1^2$$

$$= \frac{1}{3} 2\sqrt{2} - \frac{1}{3}$$

$$= \frac{1}{3}(2\sqrt{2} - 1)$$

A. $2\sqrt{3}$

B. $\frac{1}{3}(2\sqrt{2}-1)$

C. $\frac{\sqrt{3}}{3}$

D. $\frac{1}{3}(\sqrt{3}-1)$

E. $(3\sqrt{2}-1)$

25. In the Taylor series of $f(x) = \dfrac{1}{x}$ about $a = 2$, the coefficient of $(x-2)^3$ is

$$f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f'''(2) = -6 \cdot 2^{-4}$

$$\frac{f'''(2)}{3!} = \frac{-6}{6} \frac{1}{16} = -\frac{1}{16}$$

A. $-\frac{1}{3!}$

B. $\frac{3}{32}$

C. $-\frac{1}{20}$

D. $-\frac{1}{16}$

E. $-\frac{1}{40}$