Solns - Ex 1 - Fall 2015 - MA 261 - version 01.

Name:

PROBLEM 1: The angle between the planes given by the equations

$$x + y = 2 \text{ and } x + y + \sqrt{2}z = \sqrt{6}$$

is

A. $\frac{\pi}{2}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{6}$

D. π

E. $\frac{\pi}{3}$

$\vec{n}_1 = \langle 1, 1, 0 \rangle$, $\vec{n}_2 = \langle 1, 1, \sqrt{2} \rangle$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{1+1+0}{\sqrt{2} \cdot \sqrt{4}}$$

$$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

PROBLEM 2: a) Consider the following two curves $r_1(t) = \langle t, t^2, t^3 \rangle$ and $r_2(t) = (-1 + 3t, 1 + 3t, -1 + 9t)$. The curves have

A. 2 intersection points and no points of collision.

B. 1 intersection point, which is a point of collision.

C. 2 intersection points, one of which is a point of collision.

D. 1 intersection point and no points of collision.

E. 2 intersection points which both are points of collision.

Solve $\vec{r}_1(s) = \vec{r}_2(t)$,

ie $\underbrace{s = -1 + 3t}_{(1)}$, $\underbrace{s^2 = 1 + 3t}_{(2)}$, $\underbrace{s^3 = -1 + 9t}_{(3)}$

$$(1) \,\&\, (2) \Rightarrow 1 + 3t = s^2 = (3t-1)^2$$
$$= 9t^2 - 6t + 1$$
$$\Rightarrow 0 = 9t^2 - 9t = 9t(t-1)$$
$$\Rightarrow t = 0 \text{ or } t = 1$$

$t = 0 \Rightarrow s = -1 \Rightarrow$ intersection @ $(-1, 1, -1)$

$t = 1 \Rightarrow s = 2 \Rightarrow$ intersection @ $(2, 4, 8)$

PROBLEM 3: Find the length of the curve given by

$$r(\vec{t}) = \langle 2t, 4\sqrt{t}, \ln t\rangle$$

for $1 \le t \le e$.

$\vec{r}'(t) = \langle 2, 2t^{-1/2}, \frac{1}{t}\rangle$

$|\vec{r}'(t)| = \sqrt{4 + \frac{4}{t} + \frac{1}{t^2}} = \sqrt{\frac{4t^2+4t+1}{t^2}}$

A. $e - 1$

B. $2e + 1$

C. $e + 1$

D. $2e - 1$

E. $4e - 3$

$= \frac{2t+1}{t} = 2 + \frac{1}{t}$.

$L = \int_1^e (2 + \frac{1}{t})\,dt = (2t + \ln t)\Big|_1^e$

$= (2e+1) - (2)$

$= 2e - 1$

PROBLEM 4: A particle is moving with acceleration $t\vec{j} + \vec{k}$. If the velocity at time $t = 1$ is $\vec{v}(1) = \vec{i} - \frac{1}{2}\vec{j}$, what is the velocity at time $t = 0$?

$\vec{a} = \vec{v}' = t\vec{j} + \vec{k}$, $\vec{v}(1) = \vec{i} - \frac{1}{2}\vec{j}$

A. $\vec{i} - \vec{j} - \vec{k}$

B. $\vec{i} - \vec{j} + 2\vec{k}$

C. $\vec{i} - \frac{1}{2}\vec{j} - \vec{k}$

D. $\vec{i} + \frac{1}{2}\vec{j} - \vec{k}$

E. $-\vec{i} + \vec{j}$

$\Rightarrow \vec{v}(t) = \frac{t^2}{2}\vec{j} + t\vec{k} + \vec{c}$

and $\vec{v}(1) = \frac{1}{2}\vec{j} + \vec{k} + \vec{c} = \vec{i} - \frac{1}{2}\vec{j}$

$\Rightarrow \vec{c} = \vec{i} - \vec{j} - \vec{k}$

$\Rightarrow \vec{v}(0) = \frac{t^2}{2}j + tk + \vec{c}\Big|_{t=0}$

$= \vec{c} = \vec{i} - \vec{j} - \vec{k}$

PROBLEM 5: The function

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

A. is continuous on $\mathbb{R}^2$.
B. is not well defined at $(0,0)$, since it evaluates to $\frac{0}{0}$.
C. has limit 0 at $(0,0)$ along the diagonal $x = y$.
D. has limit 2 at $(1,1)$.
E. is continuous on $\mathbb{R}^2 \setminus (0,0)$.

Along the diagonal $y = x$, the expression $\frac{2x\,x}{x^2+x^2}$ has limit 1, which does not equal $f(0,0)$. Hence, f is not continuous at $(0,0)$. Rational functions are continuous on their domain.

PROBLEM 6: Consider the function

$$f(x,y) = 3x^2 - 2y^2 - 2x + 3xy$$

At the point $(1,1,2)$.

A. The slope of the tangent line to the curve of intersection of the graph of f and $x = 1$ is positive.
B. The gradient of f is $\langle -7, -1, 1 \rangle$.
C. The slope of the tangent line to the curve of intersection of the graph of f and $y = 1$ is positive.
D. The partial derivatives vanish.
E. The tangent plane is $z - 2 = 5(x-1) + 6(y-2)$.

C is true because the slope of the tang. line to the curve of intersection of the graph of f and $y=1$ at $(1,1,2)$ $= f_x(1,1) = 6x-2+3y\big|_{(1,1)} = 6-2+3 = 7 > 0$

PROBLEM 7: Consider the function

$$f(x,y,z) = xyz$$

which of the following is true.

(1) $df = xdx + ydy + zdz$
(2) Its linear approximation is the tangent plane.
(3) If $\Delta x = \Delta y = \Delta z = 0.2$ then the error estimated by using differentials at $(1,2,1)$ is 1
(4) Its gradient is $\langle yz, xz, xy \rangle$
(5) Its linear approximation at $(1,1,1)$ is $L(x,y,z) = x+y+z-2$.

~~A.~~ 1,2,3 are true.
B. 3,4,5 are true.
~~C.~~ all are true.
~~D.~~ none is true.
E. only 4 is true.

$df = f_x dx + f_y dy + f_z dz$
$= yz dx + xz dy + xy dz$
So (1) is False and (4) is true
(2) is false.

(5) is true $\because L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) = 1 + 1\cdot(x-1) + 1(y-1) + 1(z-1) = x+y+z+1-3$
$= x+y+z-2$

PROBLEM 8: Consider the function

$$g(s,t) = f(t\sin(\frac{\pi}{2}s), st^2)$$

with f differentiable. Use the table of values to calculate $g_t(1,2)$

	f	g	f_x	f_y
$(1,2)$	π	5	3	2π
$(2,4)$	5	π	2	4

Let $x = t\sin(\frac{\pi}{2}s)$ & $y = st^2$.

A. $3 + 8\pi$
B. $15 + 2\pi^2$
C. 9
D. 18
E. $2\pi + 20$

$g_t = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$.

When $s=1, t=2$: $\boxed{x} = t\sin(\frac{\pi}{2}s) = 2\cdot 1 = \boxed{2}$,
and $\boxed{y} = st^2 = 1\cdot 4 = \boxed{4}$.

So $g_t(1,2) = f_x(2,4)\cdot \sin(\frac{\pi}{2}s)\Big|_{(1,2)}$
$+ f_y(2,4)\cdot 2st\Big|_{(1,2)}$
$= 2\cdot 1 + 4\cdot 4 = 18$

PROBLEM 9: Find the maximum rate of change of the function

$$f(x,y,z) = x^2y - 2yz + z^2x$$

at the point $(1, 1, -1)$.

A. $\sqrt{52}$

B. $\sqrt{44}$

C. $2\sqrt{10}$

D. $\sqrt{34}$

E. $2\sqrt{34}$

$\vec{\nabla f} = \langle 2xy+z^2, x^2-2z, -2y+2xz \rangle \big|_{(1,1,-1)}$

$= \langle 3, 3, -4 \rangle$

$|\vec{\nabla f}| = \sqrt{9+9+16} = \sqrt{34}$

PROBLEM 10: Find all critical points of the function

$$f(x,y) = x^3 + 2xy - 2y^2 - 10x$$

and classify them.

A. $(2,-1), (\frac{5}{3}, \frac{5}{6})$ one min, one max.

B. $(-1,-2), (\frac{5}{6}, \frac{5}{3})$ one min, one saddle.

C. $(-2,-1), (\frac{5}{3}, \frac{5}{6})$ one min, one saddle.

D. $(-1,-2), (\frac{5}{6}, \frac{5}{3})$ one max, one saddle.

E. $(-2,-1), (\frac{5}{3}, \frac{5}{6})$ one max, one saddle.

$0 = f_x = 3x^2 + 2y - 10$

$0 = f_y = 2x - 4y$

$\Rightarrow \boxed{x = 2y}$

$\Rightarrow 3 \cdot 4y^2 + 2y - 10 = 0$

$\Rightarrow 6y^2 + y - 5 = 0.$

So $y = \frac{-1 \pm \sqrt{1+120}}{12} = \frac{-1 \pm 11}{12} = -1$ or $\frac{10}{12} = \frac{5}{6}$

$\Rightarrow$ critical pts $(-2,-1)$ and $(\frac{5}{3}, \frac{5}{6})$

$f_{xx} = 6x$, $f_{xy} = 2$, $f_{yy} = -4$

$\Rightarrow D = f_{xx}f_{yy} - (f_{xy})^2 = -24x - 4$

$D(-2,-1) = 48 - 4 > 0$. Also $f_{xx}(-2,-1) = -12 < 0$.

$\Rightarrow$ local max @ $(-2,-1)$. So

$D(\frac{5}{3}, \frac{5}{6}) = (-24)(\frac{5}{3}) - 4 = -40 - 4 < 0$

$\Rightarrow$ saddle pt @ $(\frac{5}{3}, \frac{5}{6})$