PROBLEM 1: The angle between the planes given by the equations

\[ x + y = 2 \] and \[ x + y + \sqrt{2}z = \sqrt{6} \]

is

A. \[ \frac{\pi}{2} \]
B. \[ \frac{\pi}{4} \]
C. \[ \frac{\pi}{6} \]
D. \[ \pi \]
E. \[ \frac{\pi}{3} \]

\[ \overrightarrow{\mathbf{n}_1} = \langle 1, 1, 0 \rangle, \quad \overrightarrow{\mathbf{n}_2} = \langle 1, 1, \sqrt{2} \rangle \]

\[ \cos \theta = \frac{\overrightarrow{\mathbf{n}_1} \cdot \overrightarrow{\mathbf{n}_2}}{|\overrightarrow{\mathbf{n}_1}| \cdot |\overrightarrow{\mathbf{n}_2}|} = \frac{1+1+0}{\sqrt{2} \cdot \sqrt{4}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}. \]

\[ \Rightarrow \theta = \frac{\pi}{4} \]

PROBLEM 2: a) Consider the following two curves \( r_1(t) = (t, t^2, t^3) \) and \( r_2(t) = (-1 + 3t, 1 + 3t, -1 + 9t) \). The curves have

A. 2 intersection points and no points of collision.
B. 1 intersection point, which is a point of collision.
C. 2 intersection points, one of which is a point of collision.
D. 1 intersection point and no points of collision.
E. 2 intersection points which both are points of collision.

Solve \( \overrightarrow{r_1}(s) = \overrightarrow{r_2}(t) \),

\[ \begin{align*}
\text{i.e.} \quad s &= -1 + 3t, \quad s^2 &= 1 + 3t, \quad s^3 &= -1 + 9t \\
(1) & \quad (2) & \quad (3)
\end{align*} \]

\[ (1) \quad \& \quad (2) \Rightarrow 1 + 3t = s^2 = (3t-1)^2 \]

\[ = 9t^2 - 6t + 1 \]

\[ \Rightarrow 0 = 9t^2 - 9t = 9(t^2 - t) \]

\[ \Rightarrow t = 0 \text{ or } t = 1 \]

\[ t = 0 \Rightarrow s = -1 \Rightarrow \text{intersection at } (-1, 1, -1) \]

\[ t = 1 \Rightarrow s = 2 \Rightarrow \text{intersection at } (2, 4, 8) \]
Name:

**Problem 3:** Find the length of the curve given by

\[ r(t) = (2t, 4\sqrt{t}, \ln t) \]

for \( 1 \leq t \leq e. \)

A. \( e - 1 \)  
B. \( 2e + 1 \)  
C. \( e + 1 \)  
D. \( 2e - 1 \)  
E. \( 4e - 3 \)

\[
\begin{align*}
\vec{r}'(t) &= <2, 2e^{t^{-\frac{1}{2}}}, \frac{1}{t}> \\
|\vec{r}'(t)| &= \sqrt{4 + \frac{4e}{t} + \frac{1}{t^2}} = \sqrt{\frac{4e^2 + 4e + 1}{t^2}} \\
&= \frac{2e + 1}{t^2} = 2 + \frac{1}{t} \\
L &= \int_1^e \left(2 + \frac{1}{t}\right)dt = (2t + \ln t)|_1^e \\
&= (2e + 1) - (2) \\
&= 2e - 1
\end{align*}
\]

**Problem 4:** A particle is moving with acceleration \( t\vec{j} + \vec{k} \). If the velocity at time \( t = 1 \) is \( \vec{v}(1) = \vec{i} - \frac{1}{2} \vec{j} \), what is the velocity at time \( t = 0? \)

A. \( \vec{i} - \vec{j} - \vec{k} \)  
B. \( \vec{i} - \vec{j} + 2\vec{k} \)  
C. \( \vec{i} - \frac{1}{2}\vec{j} - \vec{k} \)  
D. \( \vec{i} + \frac{1}{2}\vec{j} - \vec{k} \)  
E. \( -\vec{i} + \vec{j} \)

\[
\begin{align*}
\vec{a} &= \vec{v}' = t\vec{j} + \vec{k}, \quad \vec{v}(1) = \vec{i} - \frac{1}{2} \vec{j} \\
\Rightarrow \vec{v}(t) &= \vec{i} - \frac{1}{2} \vec{j} + t\vec{k} + \vec{c} \\
\text{and } \vec{v}(1) &= \vec{i} + \frac{1}{2} \vec{j} + \vec{c} = \vec{i} - \frac{1}{2} \vec{j} \\
\Rightarrow \vec{c} &= \vec{i} - \vec{j} - \vec{k} \\
\Rightarrow \vec{v}(0) &= \frac{1}{2} \vec{j} + t\vec{k} + \vec{c} \big|_{t=0} \\
&= \vec{c} = \vec{i} - \vec{j} - \vec{k}
\end{align*}
\]
PROBLEM 5: The function

\[ f(x, y) = \begin{cases} 
\frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases} \]

A. is continuous on \( \mathbb{R}^2 \).
B. is not well defined at \((0, 0)\), since it evaluates to \( \frac{0}{0} \).
C. has limit 0 at \((0, 0)\) along the diagonal \( x = y \).
D. has limit 2 at \((1, 1)\).
E. is continuous on \( \mathbb{R}^2 \setminus (0, 0) \).

\( E \) is true because \( f(x, y) \) is a rational function with domain \( \mathbb{R}^2 \setminus \{(0, 0)\} \), and hence \( f \) is continuous on this domain.

PROBLEM 6: Consider the function

\[ f(x, y) = 3x^2 - 2y^2 - 2x + 3xy \]

At the point \((1, 1, 2)\).

A. The slope of the tangent line to the curve of intersection of the graph of \( f \) and \( x = 1 \) is positive.
B. The gradient of \( f \) is \((-7, -1, 1)\).
C. The slope of the tangent line to the curve of intersection of the graph of \( f \) and \( y = 1 \) is positive.
D. The partial derivatives vanish.
E. The tangent plane is \( z - 2 = 5(x - 1) + 6(y - 2) \).

\( C \) is true because the slope of the tangent line to the curve of intersection of the graph of \( f \) and \( y = 1 \) at \((1,1,2)\)

\[ f_x(1,1) = 6x - 2 + 3y \bigg|_{(1,1)} = 6 - 2 + 3 = 7 > 0 \]
Name:

**Problem 7:** Consider the function

\[ f(x, y, z) = xyz \]

which of the following is true.

1. \( df = xdx + ydy + zdz \)
2. Its linear approximation is the tangent plane.
3. If \( \Delta x = \Delta y = \Delta z = 0.2 \) then the error estimated by using differentials at (1, 2, 1) is 1
4. Its gradient is \( \langle yz, xz, xy \rangle \)
5. Its linear approximation at (1, 1, 1) is \( L(x, y, z) = x + y + z - 2 \)

A. 1, 2, 3 are true.
B. 3, 4, 5 are true.
C. All are true.
D. None is true.
E. Only 4 is true.

\[ \frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy \]

\[ \begin{align*}
\Delta f &= \left. \frac{\partial f}{\partial x} \right|_{(1,1,1)} \Delta x + \left. \frac{\partial f}{\partial y} \right|_{(1,1,1)} \Delta y + \left. \frac{\partial f}{\partial z} \right|_{(1,1,1)} \Delta z \\
&= yz \Delta x + xz \Delta y + xy \Delta z \\
&= 1 
\end{align*} \]

\[ L(x, y, z) = x + y + z - 2 \]

**Problem 8:** Consider the function

\[ g(s, t) = f(t \sin \left( \frac{\pi}{2} s \right), st^2) \]

with \( f \) differentiable. Use the table of values to calculate \( g_t(1, 2) \)

\[
\begin{array}{c|ccc}
  & f & g & f_x & f_y \\
\hline
(1, 2) & \pi & 5 & 3 & 2\pi \\
(2, 4) & 5 & 2 & \pi & 4 \\
\end{array}
\]

A. \( 3 + 8\pi \)
B. \( 15 + 2\pi^2 \)
C. 9
D. 18
E. \( 2\pi + 20 \)

\[ g_t = f_x \cdot \frac{\partial}{\partial t} + f_y \cdot \frac{\partial}{\partial t} \]

When \( s = 1, t = 2 \):

\[ x = t \sin \left( \frac{\pi}{2} \right), \quad y = st^2 \]

\[ \begin{align*}
\dot{x} &= f_x \cdot \frac{\partial}{\partial t} + f_y \cdot \frac{\partial}{\partial t} \\
\dot{y} &= \frac{df}{dt} \mid_{(x,y)} \\
&= \left. \frac{\partial f}{\partial x} \right|_{(1,2)} x + \left. \frac{\partial f}{\partial y} \right|_{(1,2)} y \\
&= 2 \cdot 1 + 4 \cdot 2 = 10 \\
\end{align*} \]
PROBLEM 9: Find the maximum rate of change of the function
\[ f(x, y, z) = x^2 y - 2yz + z^2 x \]
at the point \((1, 1, -1)\).
A. \(\sqrt{52}\)
B. \(\sqrt{44}\)
C. \(2\sqrt{10}\)
D. \(\sqrt{34}\)
E. \(2\sqrt{34}\)

\[ \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle(1, 1, -1) = \langle 3, 3, -4 \rangle \]

\[ |\nabla f| = \sqrt{9+9+16} = \sqrt{34} \]

PROBLEM 10: Find all critical points of the function
\[ f(x, y) = x^3 + 2xy - 2y^2 - 10x \]
and classify them.
A. \((2, -1), (\frac{3}{2}, \frac{1}{2})\) one min, one max.
B. \((-1, -2), (\frac{3}{2}, \frac{1}{2})\) one min, one saddle.
C. \((-2, -1), (\frac{3}{2}, \frac{1}{2})\) one min, one saddle.
D. \((-1, -2), (\frac{3}{2}, \frac{1}{2})\) one max, one saddle.
E. \((-2, -1), (\frac{3}{2}, \frac{1}{2})\) one max, one saddle.

\[ 0 = f_x = 3x^2 + 2y - 10 \]
\[ 0 = f_y = 2x - 4y \]
\[ 0 \Rightarrow x = 2y \]
\[ \Rightarrow 3y^2 + 2y - 10 = 0 \]
\[ 6y^2 + y - 5 = 0 \]
\[ y = \frac{-1 \pm \sqrt{1 + 120}}{12} = \frac{-1 \pm 11}{12} = -1 \text{ or } \frac{5}{6} \]
\[ \Rightarrow \text{critical pts} (-2, -1) \text{ and } (\frac{5}{3}, \frac{5}{6}) \]
\[ f_{xx} = 6x, f_{xy} = 2, f_{yy} = -4 \]
\[ \Rightarrow D = f_{xx}f_{yy} - (f_{xy})^2 = -24 \cdot 1 + 4 = -20 < 0 \]
\[ \Rightarrow \text{local max at } (-2, -1) \]
\[ D(\frac{5}{3}, \frac{5}{6}) = -(\frac{24}{9})(\frac{5}{6}) = -\frac{100}{27} < 0 \]
\[ \Rightarrow \text{saddle pt at } (\frac{5}{3}, \frac{5}{6}) \]