SPRING 2018 EXAM I

1. Identify the surface defined by $x^2 - y^2 - 4x + z^2 = 4$.

A. $\boxed{\text{hyperboloid of one sheet}}$

B. hyperbolic paraboloid

C. hyperboloid of two sheets

D. ellipsoid

E. cone

$(x-2)^2 - y^2 + z^2 = 8$

$y=k$: $(x-2)^2 + z^2 = 8+k^2$, circle

$x=k$, $z=k$: hyperbolas

2. If L is the tangent line to the curve $\vec{\mathbf{r}}(t) = \langle 2t-1,\ t^2,\ t^2-2\rangle$ at $(3,4,2)$, find the point where L intersects the xy-plane.

$\vec{r}(t) = \langle 3,4,2\rangle \Rightarrow t=2$

A. $(2,1,0)$

B. $(1,2,0)$

C. $(2,-2,0)$

D. $\boxed{(2,2,0)}$

E. $(0,0,0)$

$\vec{r}\,'(t) = \langle 2, 2t, 2t\rangle$

$\vec{r}\,'(2) = \langle 2,4,4\rangle$

L: $\langle 3+2t, 4+4t, \underbrace{2+4t}_{0}\rangle$

$t = -\frac{1}{2}$

$\left(3+2(-\tfrac{1}{2}),\ 4+4(-\tfrac{1}{2}),\ 0\right)$

3. Let $\vec{\mathbf{v}} = \int_0^1 \left(\frac{1}{2}\vec{\imath} + 2t^3\vec{\jmath} + (t - 3t^2)\vec{\mathbf{k}}\right) dt$. Compute $|\vec{\mathbf{v}}|$.

A. 1

B. $\frac{3}{2}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

E. $\boxed{\frac{\sqrt{3}}{2}}$

$$\vec{v} = \left\langle \frac{1}{2},\ \frac{2}{4}t^4\Big|_0^1,\ \frac{1}{2}t^2 - t^3\Big|_0^1 \right\rangle$$

$$= \left\langle \frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2} \right\rangle$$

$$|\vec{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}$$

4. Find the area of the triangle with vertices at $P(2, 2, 1)$, $Q(1, -1, 2)$, and $R(0, 1, -1)$.

A. $\sqrt{5}$

B. $\boxed{\frac{3\sqrt{10}}{2}}$

C. $\frac{\sqrt{31}}{2}$

D. $2\sqrt{5}$

E. $\frac{\sqrt{69}}{2}$

$$\overrightarrow{RP} = \langle 2, 1, 2 \rangle$$

$$\overrightarrow{RQ} = \langle 1, -2, 3 \rangle$$

$$\frac{1}{2}\left|\overrightarrow{RP} \times \overrightarrow{RQ}\right| = \frac{1}{2}\left|\langle 7, -4, 5 \rangle\right| = \frac{1}{2}\sqrt{49 + 16 + 25}$$

$f > \sqrt{x^2} - x \geq 0$

5. The level curves of $f(x, y) = \sqrt{x^2 + 4y^2 + 4} - x$ are

A. hyperbolas

B. ellipses

C. $\boxed{\text{parabolas}}$

D. sometimes lines and sometimes ellipses

E. circles

$\sqrt{x^2+4y^2+4} - x = k, \quad k > 0$

$x^2+4y^2+4 = (x+k)^2$

$x^2+4y^2+4 = x^2+2xk+k^2$

$\dfrac{4y^2+4-k^2}{2k} = x$

6. Find the length of the curve:

$$\vec{\mathbf{r}}(t) = \langle 4\sin t,\ 3t,\ -4\cos t\rangle, \quad 0 \leq t \leq \frac{1}{2}.$$

A. $\dfrac{8}{3}\sinh^{-1}\left(\dfrac{3}{8}\right)$

B. $\dfrac{8}{3}\sinh^{-1}\left(\dfrac{3}{8}\right) + \dfrac{\sqrt{73}}{8}$

C. $\boxed{2.5}$

D. 5

E. 5π

$|\vec{r}\,'| = |\langle 4\cos t, 3, 4\sin t\rangle|$

$= \sqrt{4^2(\cos^2 t + \sin^2 t) + 3^2}$

$= \sqrt{16+9} = 5$

$\int_0^{1/2} 5\, dt = 5/2$

7. A particle is moving with acceleration

$$\vec{\mathbf{a}}(t) = \langle 6,\ 6t,\ 0\rangle.$$

If at time $t = 1$, the particle has position $\vec{\mathbf{r}}(1) = \langle 2, 1, 2\rangle$, and, at time $t = 0$ it has velocity $\vec{\mathbf{v}}(0) = \langle 0, 0, 1\rangle$, compute $|\vec{\mathbf{r}}(2)|$, the magnitude of the position vector at $t = 2$.

A. $2\sqrt{53}$

B. $3\sqrt{21}$

C. $\boxed{\sqrt{194}}$

D. $\sqrt{293}$

E. $\sqrt{57}$

$\vec{v}(t) = \langle 6t + C,\ 3t^2 + D,\ E\rangle$

$\vec{v}(0) = \langle 0, 0, 1\rangle \Rightarrow C = 0, D = 0, E = 1.$

$\vec{v}(t) = \langle 6t, 3t^2, 1\rangle$

$\vec{r}(t) = \langle 3t^2 + F,\ t^3 + G,\ t + H\rangle$

$\vec{r}(1) = \langle 3 + F, 1 + G, 1 + H\rangle = \langle 2, 1, 2\rangle$

$\Rightarrow F = -1,\ G = 0,\ H = 1$

$\vec{r}(t) = \langle 3t^2 - 1,\ t^3,\ t + 1\rangle$

$|\vec{r}(2)| = |\langle 11, 8, 3\rangle| = \sqrt{121 + 64 + 9}$

8. If $f(x, y) = x\sin(xy^2)$, then $f_{xy}(\pi, 1)$ is equal to

A. 4π

B. $\boxed{-4\pi}$

C. 2π

D. -2π

E. 0

$f_x = \sin(xy^2) + xy^2\cos(xy^2)$

$f_{xy} = 2xy\cos(xy^2) + 2xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$

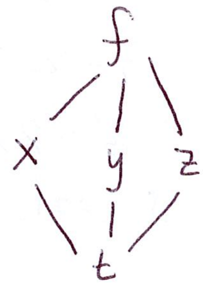
at $(\pi, 1)$:

$2\pi\cos(\pi) + 2\pi\cos(\pi) - 2\pi^2\sin(\pi)$

9. Let $f(x,y,z)$ be a function which is differentiable at $(1,1,1)$ and

$$\frac{\partial f}{\partial x}(1,1,1) = -6, \quad \frac{\partial f}{\partial y}(1,1,1) = 2, \quad \text{and} \quad \frac{\partial f}{\partial z}(1,1,1) = -1.$$

Let $\vec{\gamma}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ be the parametric equation of a differentiable curve in $\mathbb{R}^3$ and suppose $\vec{\gamma}(0) = \langle 1,1,1\rangle$ and $\frac{d\vec{\gamma}}{dt}(0) = 3\vec{\imath} - 3\vec{\jmath} + \vec{\mathbf{k}}$. We can conclude that $\frac{d}{dt}f(\vec{\gamma}(t))$ at $t = 0$ is equal to



$-6\cdot 3 + 2(-3) - 1(1)$

A. $\boxed{-25}$

B. -14

C. -13

D. -11

E. -4

10. Which of the following is an equation for the plane tangent to the surface

$$z = \tan^{-1}(x^2+y^2) \quad \text{at the point} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)?$$

Hint: $\frac{d}{du}(\tan^{-1}u) = \frac{1}{1+u^2}$

$z_x = \frac{2x}{1+(x^2+y^2)^2}$, $z_y = \frac{2y}{1+(x^2+y^2)^2}$

AT $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, \frac{\pi}{4}\right)$:

$z_y = z_x = \frac{2/\sqrt2}{1+(\frac12+\frac12)^2} = \frac{2/\sqrt2}{2} = \frac{1}{\sqrt2}$

$z - \frac{\pi}{4} = \frac{1}{\sqrt2}\left(x - \frac{1}{\sqrt2}\right) + \frac{1}{\sqrt2}\left(y - \frac{1}{\sqrt2}\right)$

$\sqrt2 z - \sqrt2\frac{\pi}{4} = x - \frac{1}{\sqrt2} + y - \frac{1}{\sqrt2} = x+y-2\left(\frac{1}{\sqrt2}\right) = x+y-\sqrt2$

$\sqrt2 - \sqrt2\frac{\pi}{4} = x+y-\sqrt2 z$

A. $x + y - \frac{1}{2}z = \sqrt{2} - \frac{\pi}{8}$

B. $x + y - 2z = \sqrt{2} - \frac{\pi}{2}$

C. $\boxed{x + y - \sqrt{2}z = \left(1 - \frac{\pi}{4}\right)\sqrt{2}}$

D. $x + y - z = \sqrt{2} - \frac{\pi}{4}$

E. $x + y - 3z = \frac{\sqrt{2}}{3} - \frac{3\pi}{4}$

11. The absolute minimum value of $f(x, y) = 2 + x^2y^2$ in the region $\frac{x^2}{2} + y^2 \le 1$ equals 2. The absolute maximum value of f in this region is

A. 4.5

B. 4

C. 3.5

D. 3

E. $\boxed{2.5}$

$f_x = 2xy^2 = 0$, $f_y = 2x^2y = 0$ $\Leftrightarrow$ $x=0$ or $y=0$, and so, $f=2$, the minimum

On boundary, $\frac{x^2}{2} + y^2 = 1$, $y^2 = 1 - \frac{x^2}{2}$, so

$f = 2 + x^2\left(1 - \frac{x^2}{2}\right) = 2 + x^2 - \frac{x^4}{2}$, $-\sqrt{2} \le x \le \sqrt{2}$

$f' = 2x - 2x^3 = 2x(1-x^2) = 2x(1-x)(1+x)$

If $x = \pm 1$, $y^2 = 1 - \frac{1}{2} = \frac{1}{2}$, so $f = 2 + (1)\left(\frac{1}{2}\right)$

12. Let $z(x, y)$ be the function implicitly defined as the solution to

$$x + y + z + \sin(xyz) = 3 + \frac{\pi}{2}$$

that satisfies $z(1, 1) = \frac{\pi}{2}$. Find $z_x(1, 1)$.

A. 1

B. $\boxed{-1}$

C. 2

D. 0

E. $\frac{3}{2}$

$\frac{\partial}{\partial x}\left[x + y + z + \sin(xyz)\right] = \frac{\partial}{\partial x}\left[3 + \frac{\pi}{2}\right]$

$1 + 0 + z_x + \cos(xyz) \cdot (yz + xyz_x) = 0$

$1 + z_x + \underbrace{\cancel{\cos\left(1 \cdot 1 \cdot \frac{\pi}{2}\right)}}_{0} \cdot (1 + z_x) = 0$

$z_x = -1$