Several different methods are often given, and we have tried to explain steps

261 Test 2 FORM A

1. A vector which points in the direction at which the function $f(x, y) = x^2 + 3xy - \frac{1}{2}y^2$ increases most rapidly at $(1, 1)$ is:

$\nabla f = (2x+3y, 3x-y)$

$\nabla f(1,1) = (5, 2)$ (direction of most rapid increase)

A. $\mathbf{i}+\mathbf{j}$

B. $5\mathbf{i}-\mathbf{j}$

C. $5\mathbf{i}+2\mathbf{j}$

D. $2\mathbf{i}-5\mathbf{j}$

E. $\mathbf{i}-\mathbf{j}$

MATH 261

EXAM 2 SOLUTIONS

2. Evaluate $\iiint_E x\,dV$ where $E = \{(x,y,z) | 0 \leq x \leq 1,\ 0 \leq y \leq 2x,\ y \leq z \leq 3x\}$.

information suggests order of integration:

$\iiint dz\,dy\,dx$

A. 0

B. 1

C. -1

D. 3

E. -3

$$\int_0^1 \int_0^{2x} \int_y^{3x} x\,dz\,dy\,dx = \int_0^1 x \int_0^{2x} (3x - y)\,dy\,dx$$

$$= \int_0^1 x \left(3xy - \frac{1}{2}y^2 \Big|_0^{2x}\right) dx$$

$$= \int_0^1 x\left(6x^2 - \frac{1}{2}\cdot 4x^2\right) dx = \int_0^1 6x^3 - 2x^3\,dx = \int_0^1 4x^3\,dx = \frac{4}{4} = 1$$

3. If $f(x,y) = x^2 + y^2 + 2y + 1$, find the absolute maximum and absolute minimum of f on the region $\underbrace{\{(x,y) : x^2 + y^2 \le 4\}}_{D}$

A. 9 and 1

B. 8 and 1

C. 9 and -1

D. 8 and 0 ← min

E. 9 and 0

inside D look at critical points

$\nabla f = (2x, 2y+2)$

So $\nabla f = 0$ at $\boxed{(0,-1)}$, $f = 0+1-2+1 = 0$ there

on boundary there are several ways

(a) $x^2+y^2 = 4$ on bdy. So $f(x,y) = 4+2y+1 = 5+2y$ there. So on bdy minimum is $5+2(-2)$ [when $y=-2$, $x=0$], max is $5+2(2) = 9$ ← max

(b) Using Lagrange multipliers — we extremize $\underbrace{x^2+y^2+2y+1}_{f}$ subject to $\underbrace{x^2+y^2}_{g} = 4$

$\nabla f = (2x, 2y+2)$ $\quad \nabla g = (2x, 2y)$

$\nabla f = \lambda \nabla g$; $(x, y+1) = \lambda(x, y)$

Possible only if $x = 0^*$ (so $y = \pm 2$). We just saw:

$f(0,-2) = -2$, $f(0,+2) = 9$

(c) (for Drasin lecture): $\nabla f \parallel \nabla g$

$(x, y+1) \parallel (x, y)$

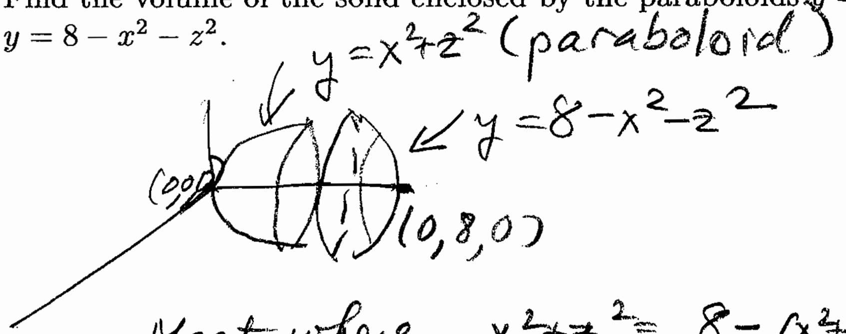
again $x = 0$.

Summary: We have to evaluate f at $(0,-1)$ (critical pt inside) and at $(0,-2)$ and $(0,2)$ (on boundary)

$*$ $x = \lambda x$, $y+1 = \lambda y$ / so $x = 0$ (see above) or $\lambda = 1$; $y(1-\lambda) = -1$, but if $\lambda = 1$ this is impossible

(D) Complete square: 3 $f(x,y) = x^2 + (y+1)^2$ = square of distance from $(0,-1)$. Clearly smallest at $(0,-1)$ and largest at $(0,2)$

4. Find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.



A. 2π

B. 4π

C. 8π

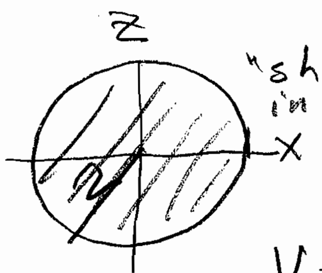
D. 16π

E. 32π

Meet where $x^2+z^2 = 8-(x^2+z^2)$

$$2(x^2+z^2) = 8$$

$$(x^2+z^2) = 4$$



"shadow" in x and z

So

$$V = \underbrace{\iint}_{x^2+z^2\le 4} \int_{x^2+z^2}^{8-(x^2+z^2)} dy\, dz\, dx$$

$$= \iint_{x^2+z^2\le 4} 8 - 2(x^2+z^2)\, dz\, dx$$

Use polar coordinates; $x = r\cos\theta$ $z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 (8-2r^2)\, r\, dr\, d\theta = 2\pi\left(4r^2 - \tfrac{1}{2}r^4\Big|_0^2\right)$$

$$= 2\pi(16-8) = 16\pi$$

5. If $(\overline{x}, \overline{y})$ is the center of mass of a thin metal plate of uniform density defined by $x^2 \leq y \leq 4$ and $x \geq 0$, then $\overline{x}$ is:

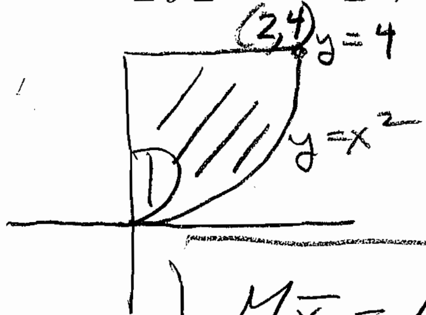
A. $\frac{3}{4}$

~~B.~~ $\frac{1}{2}$

C. $\frac{5}{4}$

D. $\frac{3}{2}$

E. $\frac{9}{10}$



$$\boxed{M\overline{x} = M_y = \iint_D x\,dy\,dx}$$

$$M = \int_0^2\int_{x^2}^4 dy\,dx = \int_0^2 4-x^2\,dx$$

$$= 8-\frac{8}{3} = \frac{16}{3}$$

$$M_y = \int_0^2 x\int_{x^2}^4 dy\,dx = \int_0^2 4x-x^3\,dx$$

$$= 2x^2-\frac{1}{4}x^4\Big|_0^2 = 8-4 = 4$$

So $\overline{x} = \frac{M_y}{M} = \frac{4}{16/3} = \frac{12}{16} = \frac{3}{4}$

6. The double integral $I = \int_0^2 \int_0^{y/2} 2x^2 y e^{xy^2}\, dx dy$ equals which of the following?
(Hint: interchange the order of integration.)

A. $\int_0^2 e^{4x} - e^{4x^2}\, dx$

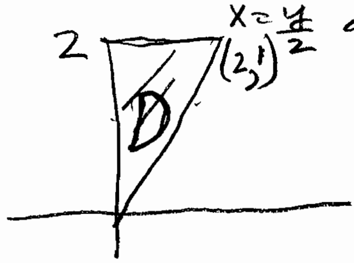
B. $\int_0^{2x} x^2(e^{4x} - e^{4x^3})dx$

C. $\int_0^1 x(e^{4x} - e^{4x^3})dx$

D. $\int_0^1 e^{4x^3} - 1 dx$

E. $\int_0^1 e^{4x} - e^{4x^2}\, dx$

Let D be region:



$x = \frac{y}{2}$ or $y = 2x$

$$I = \int_0^1 \int_{2x}^2 2x^2 y e^{xy^2} dy\, dx$$

$u = xy^2 \quad du = 2xy\, dy$

$$= \int_0^1 x \int_{2x}^2 \underbrace{2xy}_{du} \underbrace{e^{xy^2}}_{e^u} dy\, dx \qquad \left(\int e^u du = e^u\right)$$

this is u

$$= \int_0^1 x e^{xy^2}\Big|_{2x}^{2} dx = \int_0^1 x(e^{4x} - e^{4x^3})dx$$

7. Suppose (0,0) and (1,1) are critical points of $f(x,y)$, and that $f_{xx}(0,0) > 0$, $f_{yy}(0,0) > 0$, $f_{xx}(1,1) > 0$, and $f_{yy}(1,1) < 0$.
Which of the following statements must be true?

A. $(0,0)$ **is not** a local maximum and $(1,1)$ **is** a local maximum

B. $(0,0)$ **is not** a local minimum and $(1,1)$ **is** a local maximum

C. $(0,0)$ **is not** a local maximum and $(1,1)$ **is** a local minimum

D. $(0,0)$ **is not** a local minimum and $(1,1)$ **is** a saddle point

E. $(0,0)$ **is not** a local maximum and $(1,1)$ **is** a saddle point

Couple of ways.

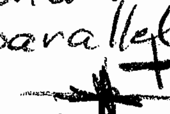
I If we use "Second derivatives test" from cover page, then we know nothing about f_{xy} at either point. But

$f_{xx}f_{yy}$ is <u>positive</u> at (0,0) (A)

is <u>negative</u> at (1,1) (B)

So if we look at (B), we know already that at (1,1) we can only have $f_{xx}f_{yy} - f_{xy}^2 < 0$ since there is a − in front of f_{xy}. So (1,1) <u>is a saddle</u>. Only D or E are possible. But by condition (A) on cover, since $f_{xx}(0,0) > 0$ we could NOT have a local maximum [if $f_{xx}f_{yy} - f_{xy}^2 > 0$ we would know we have a local minimum, but problem does not give enough information]

II Look near (0,0) and (1,1) on lines on which x changes and then y changes — lines parallel to the x and y axes.



<u>At (0,0)</u> we have a local minimum in x and y on these axes [this is what we learn in first-semester calculus], So it can't be a local max at (0,0).
at (1,1) we are told there is a local min parallel to the x axis, and a local max relative to y. So has to be saddle.

8. If $x^3 + yz = z^3 x$, then $\dfrac{\partial z}{\partial x}$ at $x = 1,\ y = 0,\ z = 1$ equals

A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $-\frac{2}{3}$

D. 0

E. $-\frac{1}{3}$

$F(x,y,z)$: $x^3 + yz - z^3 x = 0$.

Differentiate with respect to x, remembering that x, y are independent variables, but z depends on x (that is why we are asked for z_x),

$$3x^2 + y\frac{\partial z}{\partial x} - 3z^2x\frac{\partial z}{\partial x} - z^3 = 0$$

We are at $(1, 0, 1)$ So the equation becomes

$$3 + 0 - 3\frac{\partial z}{\partial x} - 1 = 0$$

$$3\frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial x} = \frac{2}{3}$$

9. Compute the area of that part of the surface $z = x^2 + y^2$ lying above $\underbrace{\{(x,y)\colon x^2+y^2 \le 4, y \ge 0\}}_{\text{shadow}}$.

A. $\frac{\pi}{3}(5^{3/2}-1)$

B. $\frac{\pi}{6}(5^{3/2}-1)$

C. $\frac{\pi}{12}(9^{3/2}-1)$

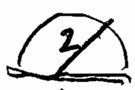
D. $\frac{\pi}{12}((17)^{3/2}-1)$

E. $\frac{9\pi}{4}$

Formula for Surface area:

$$S = \iint_{\text{shadow}} \sqrt{1+z_x^2+z_y^2}$$

$$= \iint \sqrt{1+4x^2+4y^2}\,dx\,dy$$



Use polar coordinates

$$S = \int_0^{\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1+4r^2$

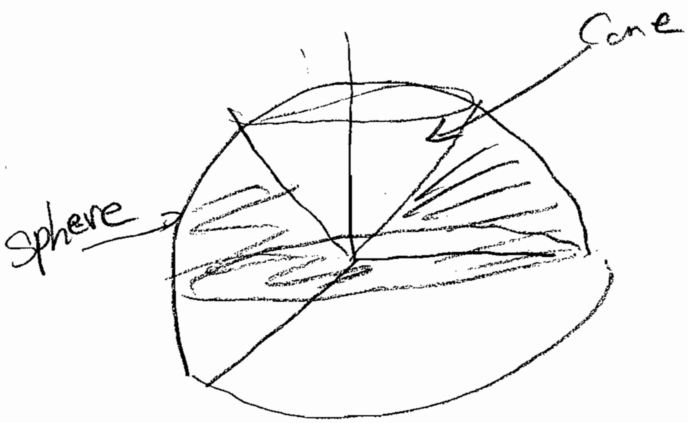
$du = 8r\,dr$

$$= \int_0^{\pi} \frac{1}{8}\int_1^{17} u^{1/2}\,du = \int_0^{\pi} \frac{1}{8}\cdot\frac{2}{3}\,u^{3/2}\Big|_1^{17} d\theta$$

$$= \frac{\pi}{8}\cdot\frac{2}{3}\left((17)^{3/2}-1\right) = \frac{\pi}{12}\left((17)^{3/2}-1\right)$$

clue!

10. Which iterated integral is equal to the volume of the solid bounded by the xy–plane, the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and the cone $z = \sqrt{x^2 + y^2}$



A. $\int_0^{2\pi} \int_0^{\pi/4} \int_{-1}^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$

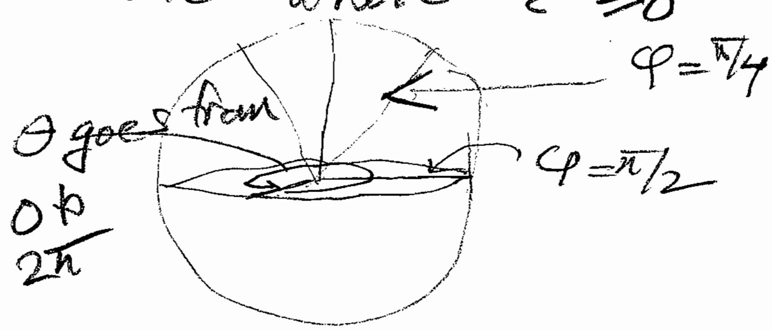
B. $\int_0^{2\pi} \int_0^{\pi/2} \int_0^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$

C. $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$

D. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$

E. $\int_0^{\pi} \int_0^{\pi/4} \int_{-1}^{1} \rho^2 \sin\phi \, d\rho d\phi d\theta$

the region is inside the sphere, outside the cone where $z \geq 0$



Use Lagrange $\quad \nabla f = (y, x)$ $\quad$ int $\overbrace{x^2 + 2y^2}^{g} = 1.$

(I) $\quad \nabla g = (2x, 4y)$

A. 0

B. $\frac{1}{2}\sqrt{2}$

C. $\frac{1}{2}$

D. $\frac{1}{\sqrt{2}}$

$\nabla f = \lambda \nabla g$

$(y, x) = \lambda(2x, 4y)$ or

$(*)$ $\left.\begin{array}{l} y = 2\lambda x \\ x = 4\lambda y \end{array}\right\}$ $y = 8\lambda^2 y$ so $y = 0$ (whence $f = 0$)

or $\lambda^2 = \frac{1}{8}$, $\lambda = \pm\frac{1}{2\sqrt{2}}$

So we take $\lambda = \pm\frac{1}{2\sqrt{2}}$; then first equation in $(*)$ becomes $y = 2 \pm \frac{1}{2\sqrt{2}} x$, $\boxed{y = \pm\frac{1}{\sqrt{2}} x}$. But $x^2 + 2y^2 = 1$

So $x^2 + 2 \cdot \frac{1}{2} x^2 = 1 \quad 2x^2 = 1 \quad x = \pm\frac{1}{\sqrt{2}}$

But if $y = \pm\frac{1}{\sqrt{2}} x$, then $y = \pm\frac{1}{2}$.

To maximize $f(x,y) = xy$, we take $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$; $f = \boxed{\frac{1}{2}\sqrt{2}}$

(II) $\nabla f \parallel \nabla g \quad (y, x) \parallel (2x, 4y)$. If x or $y = 0$, then $f = 0$. Otherwise

$$\frac{y}{2x} = \frac{x}{4y}, \quad 2x^2 = 4y^2, \quad x^2 = 2y^2$$

So Equation of g gives $x^2 + 2y^2 = 2x^2 = 1$

$x = \pm\frac{1}{\sqrt{2}}$. Rest as above