PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2001 Series)

Problem: Let $\{a_0, a_1, a_2, \dots\}$ be a non-zero sequence having period N, that is, $a_{k+N} = a_k$ for all $k = 0, 1, 2, \dots$. Show that

(1) $\sum_{k=0}^{\infty} a_k z^k$ is a rational function for $|z| < 1$,

(2) $\sum_{k=0}^{\infty} a_k$ diverges, but

(3) $\lim_{z\to 1-} \sum_{k=0}^{\infty} a_k z^k$ exists if and only if $\sum_{k=0}^{N-1} a_k = 0$; find the limit.

Solution (by Damir Dzhafarov (Fr. MA), edited by the Panel)

(1) Replace a_k with $a_{k(\text{mod } N)}$ for all $k = 0, 1, 2, \dots$. Then the terms of the sum may be grouped as $(a_0 z^0 + a_1 z^1 + \cdots + a_{N-1} z^{N-1}) \displaystyle\sum_{k=0}^{\infty} z^{kN}$. Since $|z| < 1$, this becomes

$$\frac{\sum_{k=0}^{N-1} a_k z^k}{1 - z^N},$$

a rational function.

(2) $\displaystyle\sum_{k=0}^{n} a_k$ diverges because $\displaystyle\lim_{k\to\infty} a_k \neq 0$.

(3) In view of (1) it suffices to find $\displaystyle\lim_{z\to 1-} \frac{\sum_{k=0}^{N-1} a_k z^k}{1 - z^N}$. The numerator of the expression within the limit approaches $\displaystyle\sum_{k=0}^{N-1} a_k$, while the denominator goes to 0. Hence, the limit exists only if $\displaystyle\sum_{k=0}^{N-1} a_k = 0$, in which case, by L'Hôpital's Rule, it becomes

$$\lim_{z\to 1-} \frac{\sum_{k=1}^{N-1} a_k k z^{k-1}}{-N z^{N-1}} = -\lim_{z\to 1-} \sum_{k=1}^{N-1} \frac{a_k k}{N} z^{k-N} = -\sum_{k=1}^{N-1} \frac{a_k k}{N}.$$

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